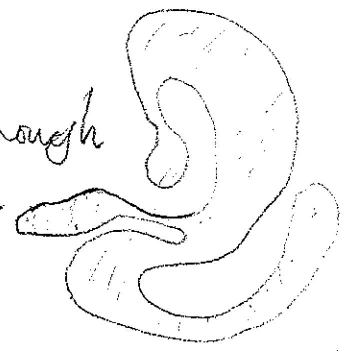
Caltech Winter 2006
Ma 109b

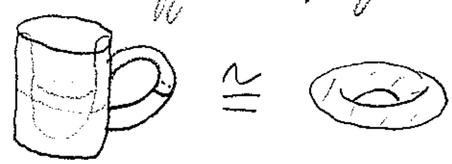
Introduction to Geometry and Topology: (1)

- Welcome! will discuss handout at end
- Note HW #1 on Handout.



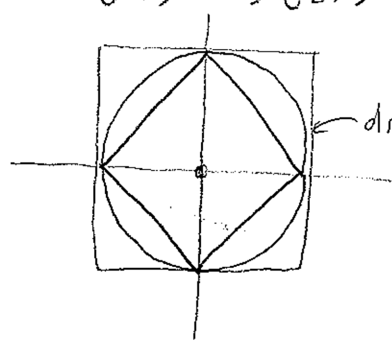
__Topology__: Study of spaces up to homeomorphism, as though made of rubber. 

__Topologist__: someone who can't tell a coffee cup from a doughnut.



__Geometry__: Study of spaces w/ distance functions (metric spaces).
[Where this dist fn is important, not just a source for the topology]

__Ex__: See above. __Ex__: $\mathbb{R}^2 = d((x_1,y_1),(x_2,y_2)) =$ Euclidean $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$
$|x_1-x_2| + |y_1-y_2|$
$\max(|x_1-x_2|, |y_1-y_2|)$


dist 1 from 0

[In both cases, our focus will be on surfaces.]

Query

__Def__: A __surface__ is a Hausdorff top space X s.t. every pt has an open nbhd homeomorphic to $\mathbb{R}^2$.

__Ex__: $\mathbb{R}^2$, , P^2, K

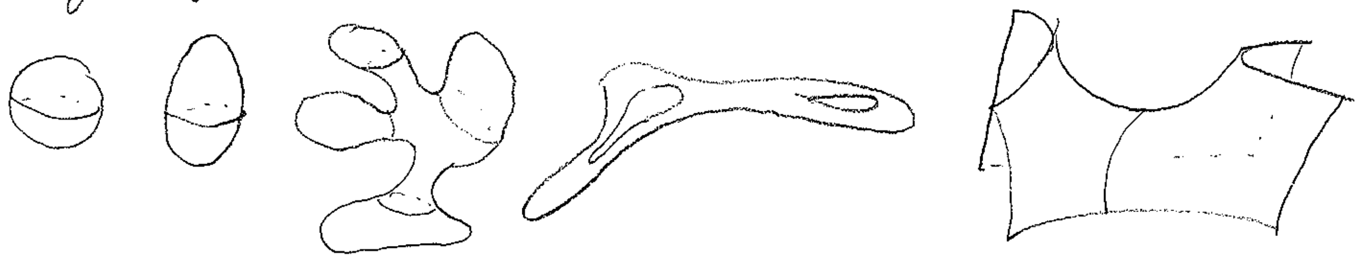
__NonEx__: $\mathbb{R}^3$,  ← almost! a surface w/ ∂.

[Discuss, history, etc. Will start off with this.]

<u>Classification</u> of <u>Surfaces</u>: Any compact surface X is homeomorphic to <u>exactly</u> one of

P^2, $K = P^2 \# P^2$, $P^2 \# P^2 \# P^2$, …

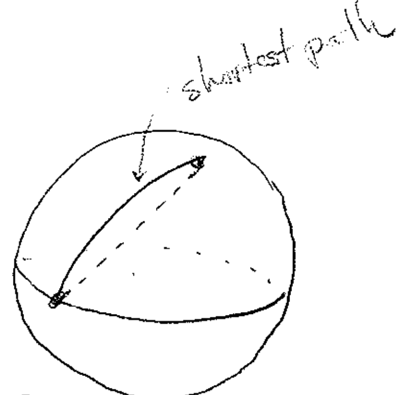
<u>Geometry of Surfaces</u>: $X^2 \subseteq \mathbb{R}^3$, smoothly embedded



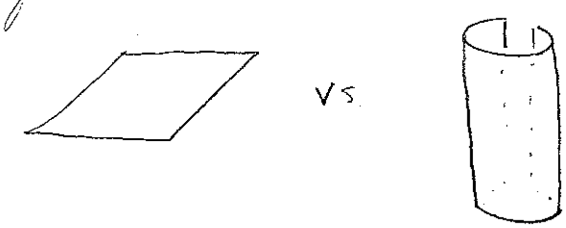
<u>Extrinsic</u>: how X sits in $\mathbb{R}^3$

vs. $\Downarrow$

<u>Intrinsic</u>: distances measured within the surface



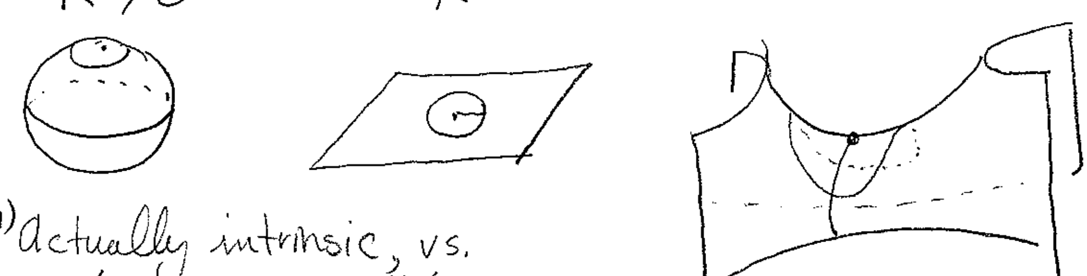
[Different surfaces w/ same intrinsic geometry



] — consider omitting

[Gaussian] Curvature:

$K > 0$ $\quad$ $K = 0$ $\quad$ $K < 0$



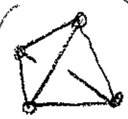
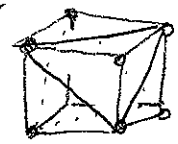
<u>Talk about</u>: 1) Actually intrinsic, vs. mean curvature in soap films

2) is a fn of the pt

Connection between: Topology and geometry:

Euler Char: T a triangulated surface

$$\chi(T) = \#\text{ of verts} - \#\text{ of edges} + \#\text{ of triangles.}$$

$\chi($  $) = 4-6+4=2$ $\quad$ $\chi($  $) = 8-18+12 = 2!$

Thm: T_1 and T_2 are triang. of the same surface S.

$$\chi(T_1) = \chi(T_2)$$

Def: $\chi(X) = \chi(T)$.

$\chi($  $) = 2$

$\chi($  $) = 0$

$\chi($  $) = -2$

Gauss-Bonnet: Suppose X is a cpt surface S in $\mathbb{R}^3$. Then

$$(\text{Average of } K)(\text{Area of } K) = 2\pi\, \chi(S)$$

Cor: Any  in $\mathbb{R}^3$ has pos curve somewhere

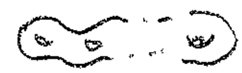
Any  in $\mathbb{R}^3$ has neg curv. somewhere.

Discuss syllab, policies, etc.

Lecture 2: • Note handout. Contains 1^{st} HW due next Wed.

Last time: Def: A surface is a top space s.t. each pt has a open nbhd $\cong \mathbb{R}^2$.

Classification Thm: A cpt surface is homeo to exactly one of

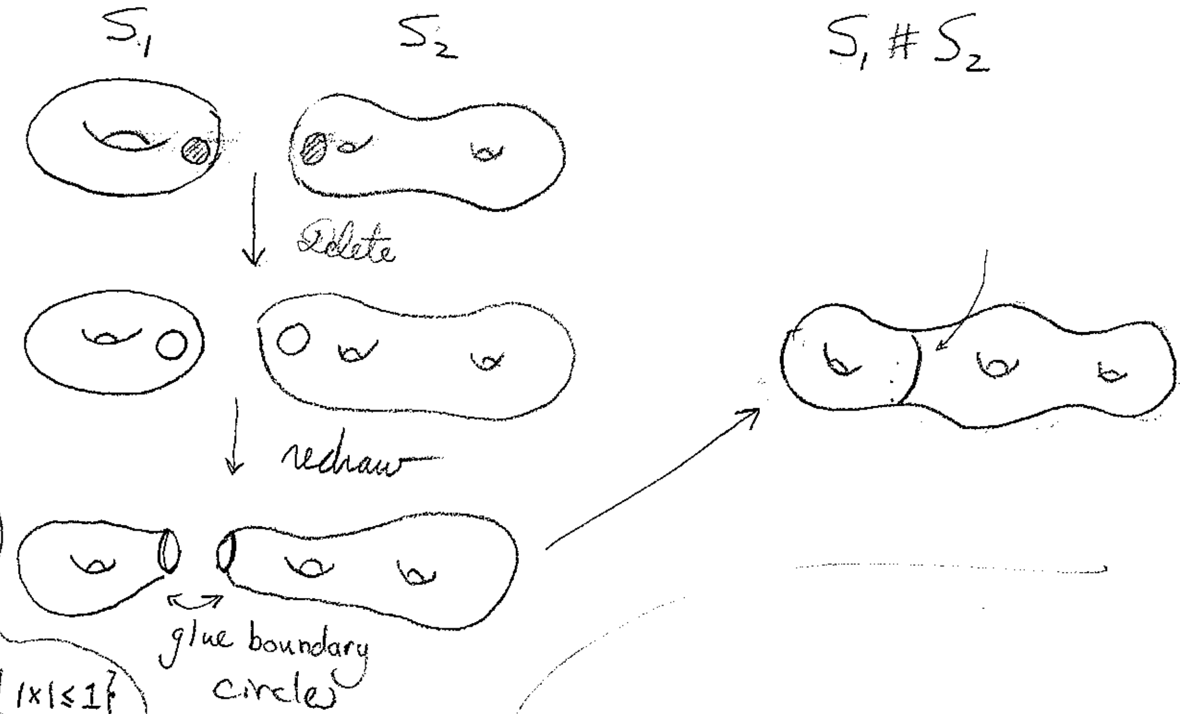
, $T =$ , $T \# T =$ , ..., $T \# \cdots \# T =$ 

P, $K = P \# P$, $P \# P \# P$, ...

Today: 1) Connected sum (#)

2) Jordan curve thm and other top. issues.

Connected sum:



Def: A chart for a surface S is an open set $\cong \mathbb{R}^2$.

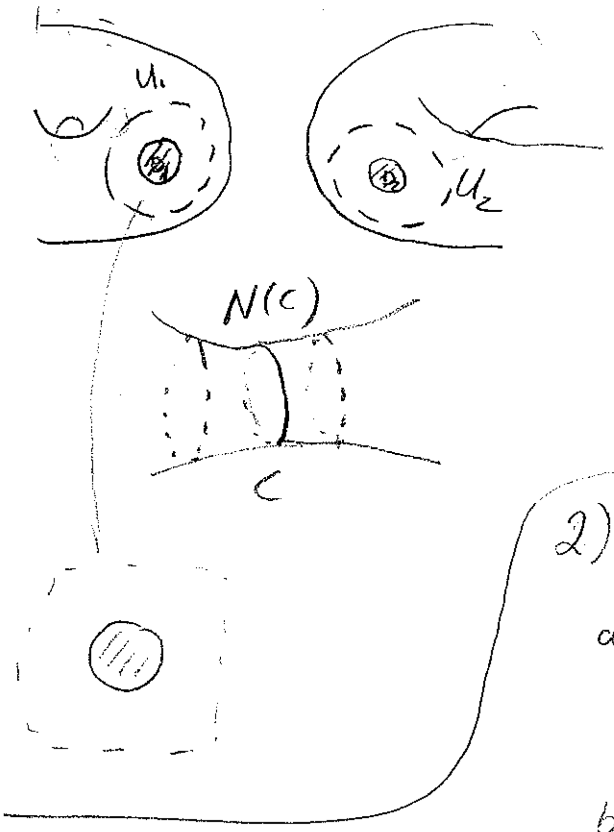
Def: A disc in S is a cpt subset D s.t. $\exists$ a chart $U \supseteq D$ where $U \xrightarrow{\cong} \mathbb{R}^2$ takes $D \longrightarrow \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$

Def: S_1, S_2 surfaces. Then $S_1 \# S_2 = (S_1 \setminus \mathring{D}_1) \cup_f (S_2 \setminus \mathring{D}_2)$ where $D_i \subseteq S_i$ is a disc and $f: \partial D_1 \longrightarrow \partial D_2$ is a homeomorphism.

Need to check: 1) $S_1 \# S_2$ is a surface

2) $S_1 \# S_2$ is indep of the choices of D_i, f.

For 1) need to check that pts along the join C have nbhds $\cong \mathbb{R}^2$ ③



$$N(C) = \underbrace{U_1 \setminus \mathring{D}_1}_{\cong S^1\times[0,\infty)} \cup_f \underbrace{U_2 \setminus \mathring{D}_2}_{\cong S^1\times[0,\infty)}$$

gluing $S^1\times\{0\}$ to $S^1\times\{1\}$ by some homeo.

$$\cong S^1\times(-\infty,\infty).$$

2) breaks into 2 issues

a) coarse: fund diff choices for f — id, reflection

$$\mathbb{Z} = \pi_1(S^1) \longrightarrow \pi_1(S^1) = \mathbb{Z}$$

$id_*(1) = 1$

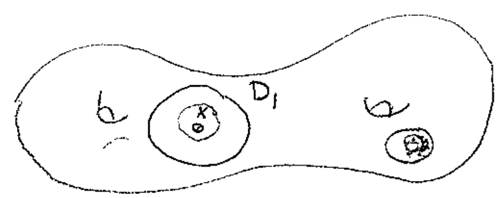
$r_*(1) = -1$

b) subtle: choices of D_i

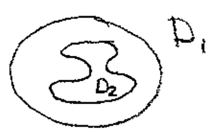
[a) is in some sense "accidental", a consequence of the dimension]

[I will deal with 2) in an odd way — I will prove the classification thus avoiding this issue.]

b) <u>Thm</u> $D_1, D_2 \subseteq S$ then
$\exists\ S \xrightarrow[\cong]{f} S$ s.t. $f(D_1) = f(D_2)$



<u>Lemma 1</u>: True if $D_2 \subseteq \mathring{D}_1$

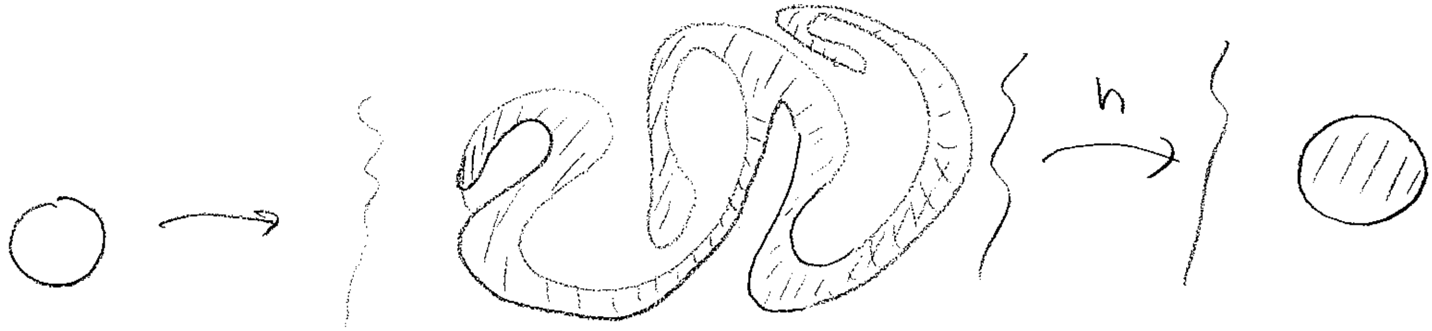


<u>Lemma 2</u>: given $x, y \in S$, $\exists$
$S \xrightarrow[f]{\cong} S$ s.t. $f(x) = f(y)$

Lemma 1 is on HW, but real work comes from:

Schönflies Thm: Let C be the image of $f: S^1 \xhookrightarrow{1-1} \mathbb{R}^2$.

Then $h: \mathbb{R}^2 \circlearrowleft \cong$ s.t. $h(C)$ is the unit circle $\{x \mid |x| = 1\}$.



Jordan Curve Thm: Any circle C as above separates $\mathbb{R}^2$ into 2 regions. [Section 5.6 of Armstrong.]

Remarks: [I was pretty non-impressed by these thms.]

1) Continuous functions are really messy. [see handout on next page]

2) Analog is not true in dim 3! $\exists\ f: S^2 \hookrightarrow \mathbb{R}^3$ s.t. some comp of $\mathbb{R}^3 \setminus f(S^2)$ is not simply connected!

See other side of handout, discuss

Thm: Any cpt surface S has a triangulation.] if time

Comment on differing w/ text.

Lecture 3: Today and Wed: proving

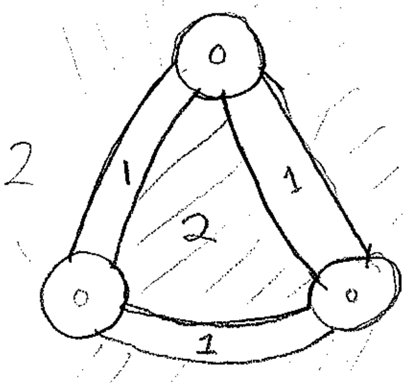
Thm: Any cpt connected surface is homeo to exactly one of

1) , ...

2) P, P#P, P#P#P, ...

[Will use diff proof than either text, say why.]

Handle Decomposition: 0) Start w/ 0-handles = D^2

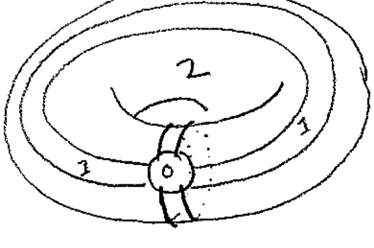


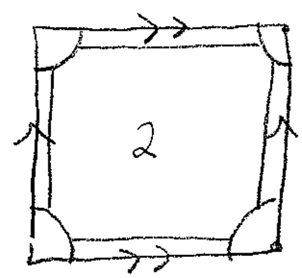
1) Glue 1-handles = $I \times I$ to the boundry of the 0-h along $\partial I \times I$ 

2) Glue 2-handles = D^2 along whole of ∂D^2  ...to every boundry comp of 0-1 handles

$\Longrightarrow$ gives a surface.

Ex:

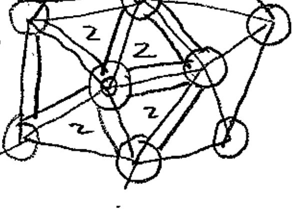




Lemma: S cpt conn surf. Then S has a handledecomp with only one 0-handle and 2-handle. [and some unknown # of 1 handles]

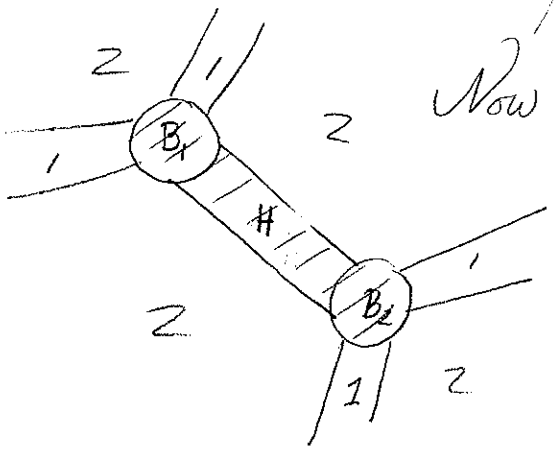
Pf: As S has a triangulation, it has a handle decomp.

Draw large



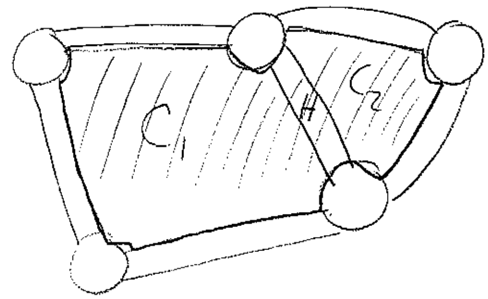
Suppose there are at least two 0-handles. Then $\exists$ two distinct 0-handles B_1, B_2 joined by a 1-handle

[ask class why. Reason: otherwise disconnected as adding 2-handle doesn't change the # of conn. comp.]



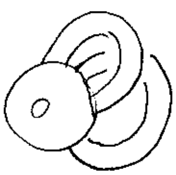
Now consider $B_1 \cup B_2 \cup H$ as a 1-handle. [Check still have a handle decomp]

Similarly if have multiple 2-handles, find two adjacent across a 1-handle and then ammalgamate



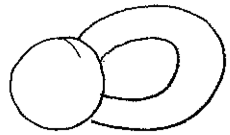
$\square$

To record such a hand. decomp. just need to draw 0 and 1 cells.

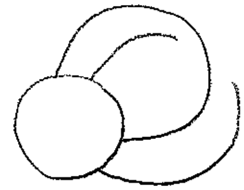


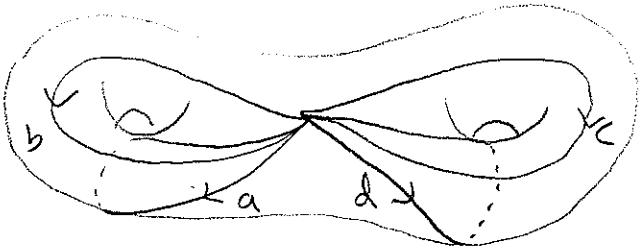
$+$ one 2-handle. $\overset{\text{[Query]}}{=} T^2$

Two kinds of 1-handle

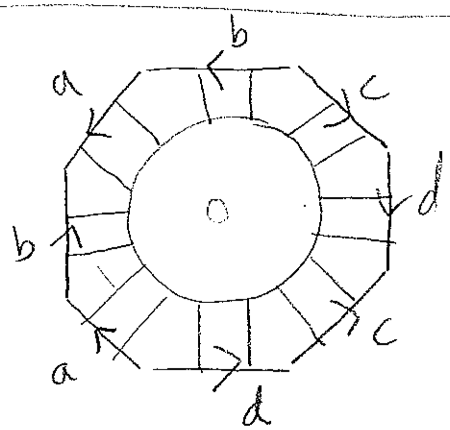


vs.

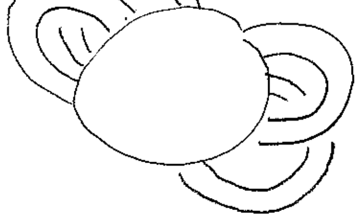




$=$



$=$

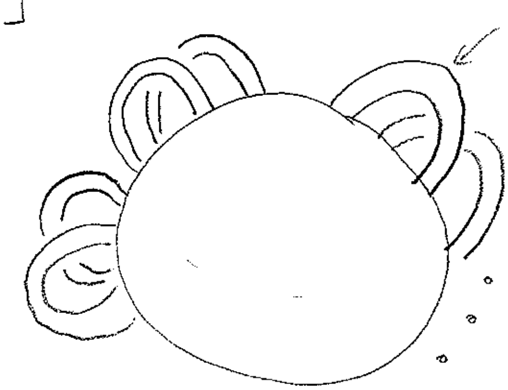


$+$ 2 handle.

Def: [To avoid # sum issue]

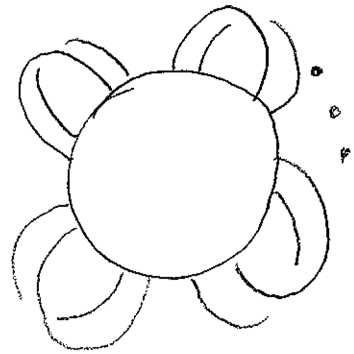
$\underbrace{T\# \cdots \# T}_{n \text{ times}} :=$

n of these.



+ 2 handle

$\underbrace{P\# \cdots \# P}_{n} =$



+ 2-handle.

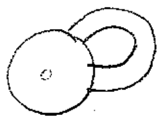
Pf of Class Thm: Consider a handle decomp of S w/ one 0 handle, one 2-handle and n 1-handles.

Orientable case: no band is twisted.

Claim: $S \cong \underbrace{T\# \cdots \# T}_{n/2}$

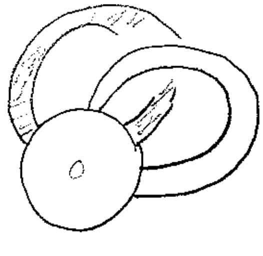
$n=0$: 0 + 2 $= S^2$

$n=1$:



← not allowed as we would have to add two 2-handles to make a surface.

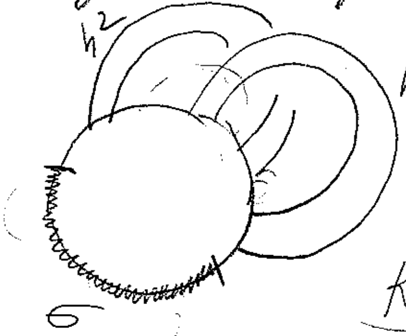
$n=2$



So we don't have the same problem as before, 2nd 1-handle must be

$= T$ ✓

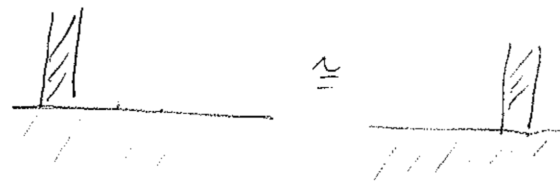
In general, suppose we have 1-handles $h^1, h^2, \dots, h^n$

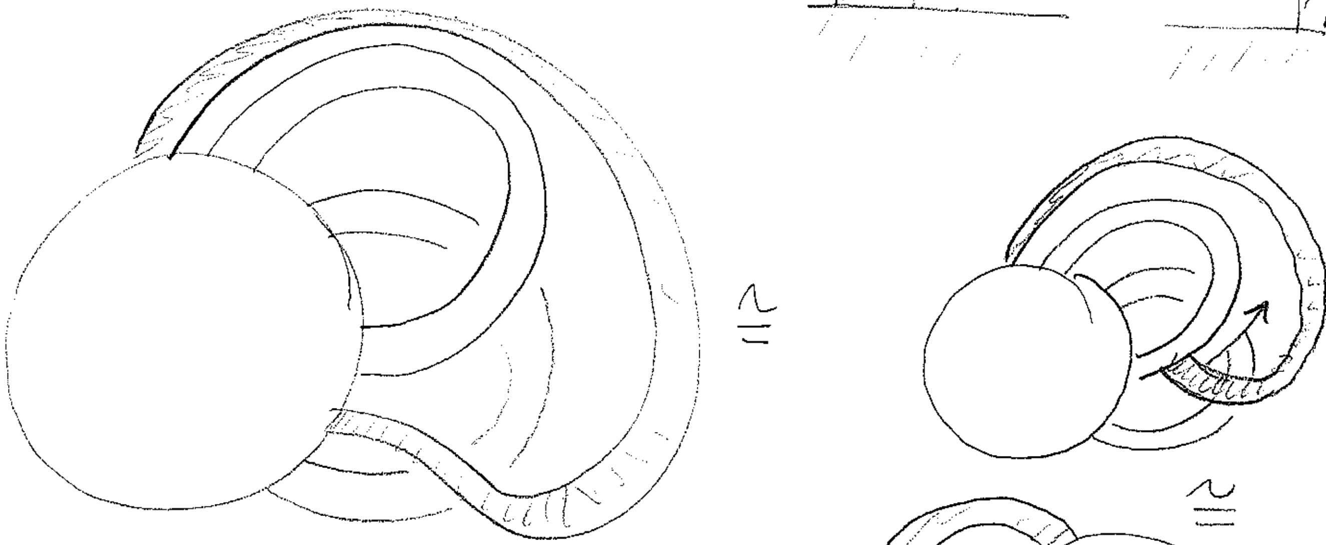


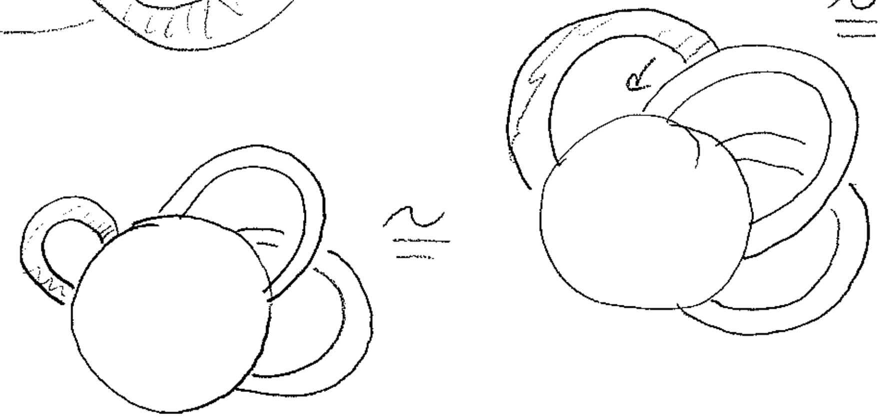
There must be some h^i, say h^2, going from one bdry comp to the other

<u>Key claim</u>: Can assume h^i for $i > 2$ are all attached out here

<u>Pf</u>: use handleslide







<u>Point</u>: the boundary of

is just a circle

Now repeat the argument on $h^3, \dots, h^n$ taking care to never leave them outside the region σ.

This concludes the orientable case. [<u>Query</u>: Abel. π_1]

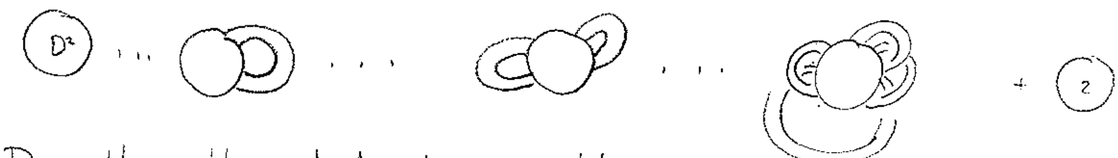
Lecture 4: On side board: Class. thm, lemma on existence of handle decomp, def of $T\#\cdots\#T$, $P\#\cdots\#P$ as handles + discs w/ ident.

Pf of Class: S a cpt conn surface. Choose a handle decomp w/ one 0-handle and one 2-handle.

Case 1: There are no twisted 1-handles.

Claim: If there are n 1-handles $h_1, \dots, h_n$ then $S \cong \underbrace{T\#\cdots\#T}_{n/2}$

$X_0 \quad X_1 = X_0 \cup h_1 \quad X_2 = X_1 \cup h_2 \quad \dots \quad X_n \qquad S = X_n \cup D^2$



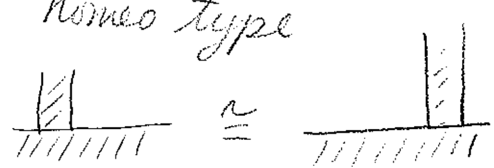
1) Doesn't matter what order we add the handles.

2) Can change handle structure of X_K using handleslides

$N(\partial X_K) \cong \partial X_K \times [0,1]$

[Query: Not a handle decomposition.]

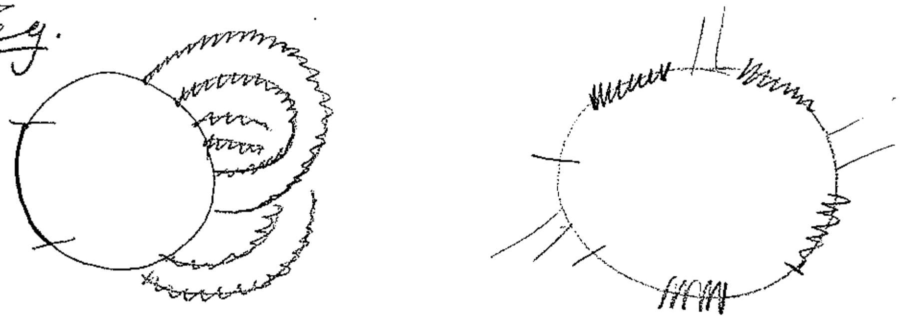
Doesn't change homeo type



[Note: May have to move $h_{K+1}, \dots, h_n$ slightly to get a handle decomp at the next stage.]

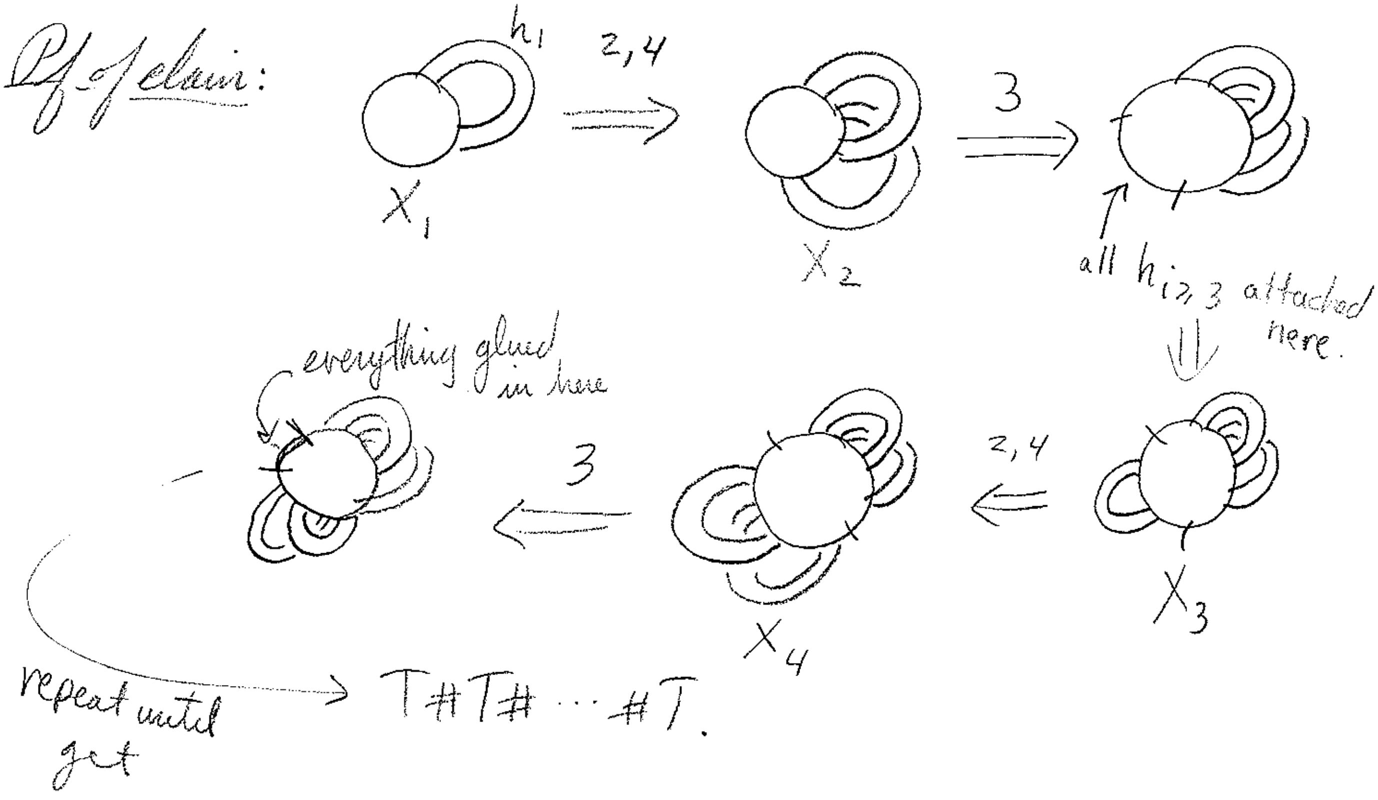
3) If ∂X_K is connected — consists of just one circle — then we can assume all remaining handles are glued to a segment of ∂X_K we get to choose

Eg.



4) If ∂X_K has two components C_1 and C_2 then at least one $h_{K+1}, \ldots, h_n$ has one end on C_1 and the other on C_2.

Pf of claim:



Claim: Suppose there are n 1-handles at least one of which is twisted. Then $S \cong \underbrace{P \# \cdots \# P}_{n}$.

Pf: HW.

To complete the proof need to show all these are distinct.

$$\pi_1(\#_n T) = \langle a_1, b_1, \ldots, a_n b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} = 1 \rangle.$$

$$\pi_1(\#_n P) = \langle a_1, \ldots, a_n \mid a_1^2 a_2^2 \cdots a_n^2 = 1 \rangle.$$

<u>Problem</u>: These a different, but how do we prove it?

<u>Def</u>: G is a gp. Then $G' =$ gp gen by $\underset{g_1 g_2 g_1^{-1} g_2^{-1}}{[g_1, g_2]}$ for all $g_1, g_2 \in G$

Set $G^{ab} = G/G'$, the abelianization of G. [why is this a normal subgp? why abelian?]

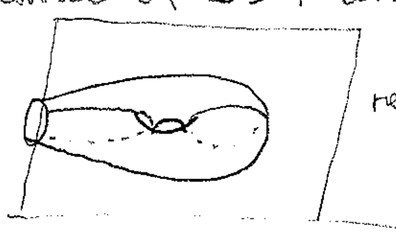
$$\pi_1(\# T^n) = \mathbb{Z}^{2n} \qquad \pi_1(\#_n P) = \mathbb{Z}^{n-1} \oplus \mathbb{Z}/2.$$

To compute, add relations so that all the gens commute.

Why is # sum well-defined?

course issue: two diff homeos of $S^1 <^{id}_{r = \text{reflection}}$

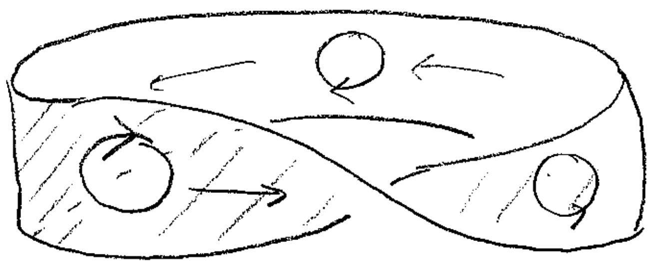
<u>Point</u>: S a compact connected surface w/ one boundry circle, then $\exists$ a homeo of $S \circlearrowleft f$ which flips the ∂ circle:



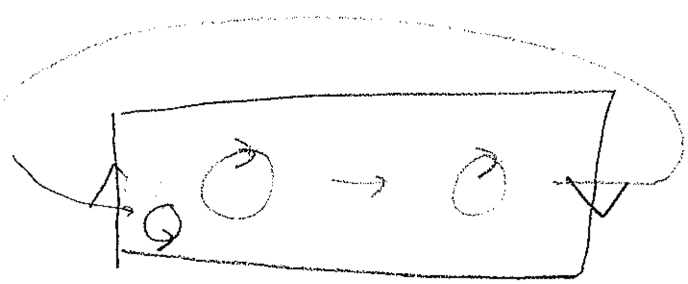
reflect. or

works for any $T \# \cdots \# T$

If surface contains a Möbius band, e.g it has a twisted handle, then just slide the boundary circle around the band



to reverse the orientation.



——— ∘ ———

Lecture 5: Smooth surfaces in $\mathbb{R}^3$

Def: If $U \subseteq \mathbb{R}^n$ then $f: U \to \mathbb{R}^m$ is <u>smooth</u> if
- U is open [need to sense of $\frac{\partial}{\partial x}$]
- all partial derivatives of f of all orders exist and are continuous.

The <u>derivative</u> of f at p is $D_p f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \cdots & \frac{\partial f_m}{\partial x_n}(p) \end{pmatrix}$.
where $f(\vec{x}) = (f_1(\vec{x}), \dots, f_m(\vec{x}))$

Gives best linear approximation to f

$$f(x) = f(x_0) + (D_{x_0} f)(x - x_0) + E(x - x_0)$$

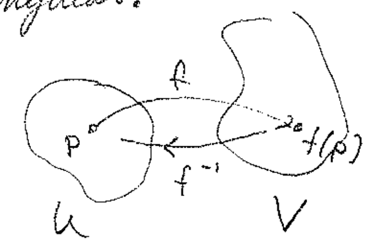
where $\exists \delta, M$ s.t. $|E(x-x_0)| \leq M |x - x_0|^2$ for all $|x - x_0| < \delta$.

Def: U, V open sets in $\mathbb{R}^n$. A fn $f: U \to V$ is a diffeomorphism if it is bijective and f, f^{-1} are both smooth. [invertible, full rank]

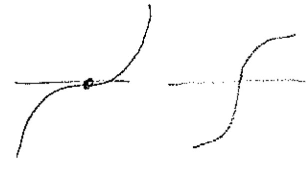
[A kind of homeomorphism]

Note: If f is a diffeo, then $\forall p \in U$, $D_p f$ is non-singular.

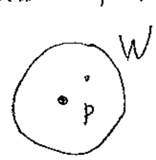
Pf: $D_{f(p)} f^{-1} \circ D_p f = D_p(f^{-1} \circ f) = D_p(Id) = I$.



Ex: A non-diffeo: $f: \mathbb{R} \circlearrowleft$ $f(x) = x^3$ a smooth homeo
$f^{-1}(x) = x^{1/3}$ not diff at 0.



<u>Inverse Function Thm</u>: $f: (U \subseteq \mathbb{R}^n) \to \mathbb{R}$ smooth.
If $p \in U$ is such that $D_p f$ is invertible
$\exists$ a open nbhd W of U such that $f(W)$ is open and $f: W \to f(W)$ is a diffeomorphism.



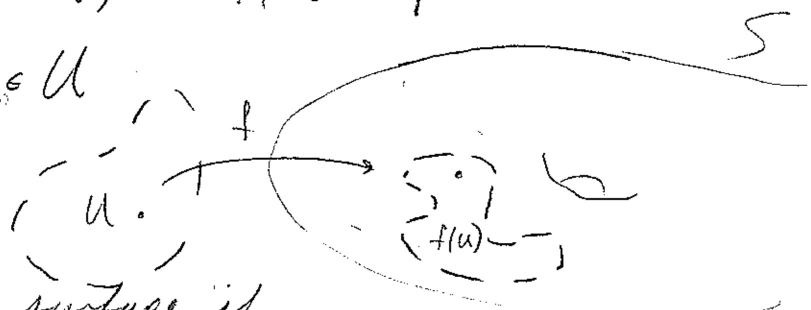
Def: $U^{\text{open}} \subseteq \mathbb{R}^2$, $f: U \to S \subseteq \mathbb{R}^3$ a smooth map.

Then f is a coordinate patch if

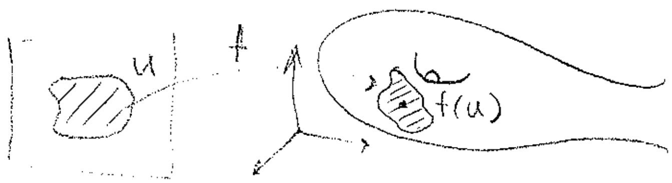
1) f is a homeo from U to $f(U)$, and $f(U)$ is open in S.

2) $D_p f$ is 1-1 for each $p \in U$

[diffeo-like, let us define a tangent plane.]



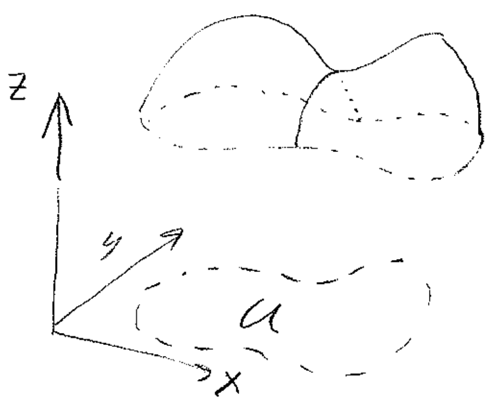
Def: $S \subseteq \mathbb{R}^3$ is a smooth surface if for each $p \in S$, there is a coordinate patch $f: U \to S$ with $p \in f(U)$



Note: such an S is also a topological surface. in the old sense [note that $U \not\cong \mathbb{R}^2$]

[Def differs from text, doesn't require it is a homeo.]

Ex: $U \overset{\text{open}}{\subseteq} \mathbb{R}^2$, $h: U \to \mathbb{R}$ smooth fn



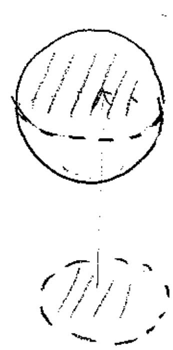
Monge Patch

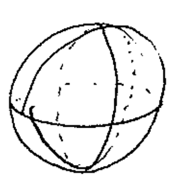
$$S = \{(x, y, h(x,y)) \mid (x,y) \in U\}$$

$f(x,y) = (x, y, h(x,y))$ a coor. patch.

$$Df = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{pmatrix}$$

Ex:





$h = \sqrt{1 - (x^2 + y^2)}$

<u>Lecture 6</u>: <u>Last time</u>: Def of smooth surface

<u>Today</u>: · Change of coor. lemma

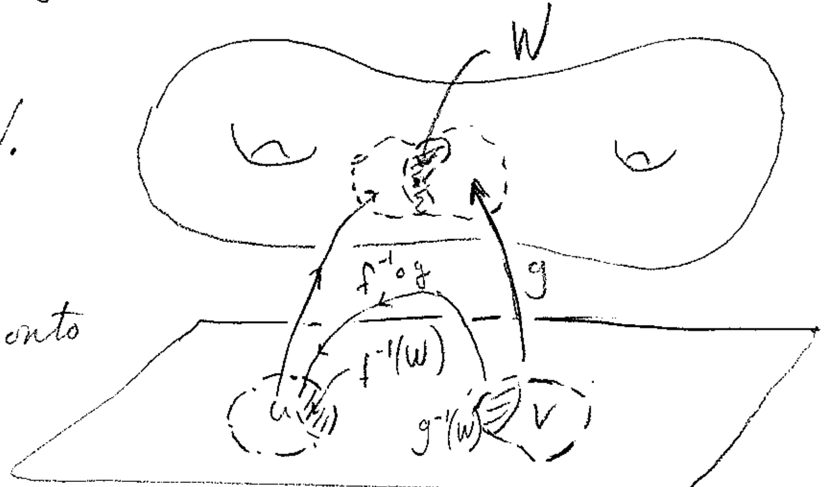
· smooth maps between surfaces, diffeo

<u>Change of Coor Lemma</u>: $S \subseteq \mathbb{R}^3$ a smooth surface. write up ahead of time

Let $f: U \to S$, $g: V \to S$ coordinate charts

Set $W = f(U) \cap f(V)$. Then $f^{-1} \circ g : g^{-1}(W) \to f^{-1}(W)$ is smooth

<u>Pf</u>: First, suppose f is a Monge chart, as in the HW.

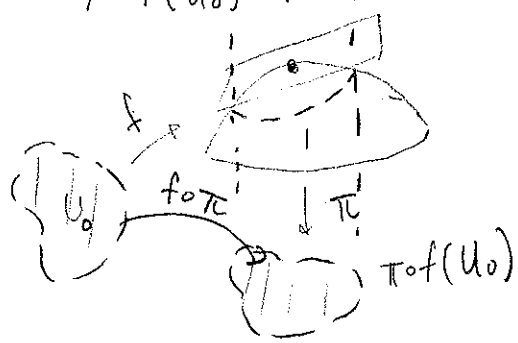


Say $f(x,y) = (x, y, h(x,y))$,

and let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be proj onto the xy plane. Then

$f^{-1} \circ g = \pi \circ g$ which is the composit of two smooth functions, hence smooth. In general, we need

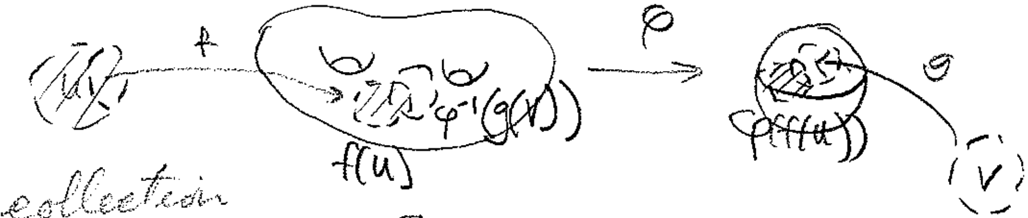
<u>Sublemma</u>: f a coor patch, p a point in $f(U)$. Then $\exists\, U_0 \overset{\text{open}}{\subseteq} U$ w/ $f(U_0) \ni p$ and a proj fn π s.t. $\pi \circ f$ is a <u>diffeo</u> on U_0.



Thus $(\pi \circ f)^{-1} \circ \pi$: open in $\mathbb{R}^3 \to U_0$ is a smooth fn and restricted to $f(U_0)$ it is just f^{-1}! This case is then just the same as before. ▨

Def: S_1, S_2 are smooth surfaces. Then $\varphi: S_1 \to S_2$ is <u>smooth</u> if $\forall$ coor patches $f: U \to S_1$, $g: V \to S_2$ we have $g^{-1} \circ \varphi \circ f : (\varphi \circ f)^{-1}(g(V)) \longrightarrow V$ is smooth.

[By change of coor lemma, only need to check cond for some collection of charts which cover S_1 and S_2]



Def: A map $\varphi: S_1 \to S_2$ is a diffeomorphism if it is a bijection and φ and φ^{-1} are smooth.

[smooth analog of homeo]:

Q: Is every topological surface a smooth surface in $\mathbb{R}^3$?

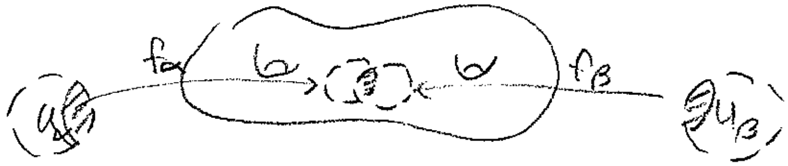
A: [Query] No. After all some don't even embed topologically in $\mathbb{R}^3$, e.g. P or K. [But this is a silly reason could just use $\mathbb{R}^4$]

<u>Abstract Smooth Surface:</u> is a topological surface S with a collection of homeos $f_\alpha : (U_\alpha \overset{\text{open}}{\subseteq} \mathbb{R}^2) \longrightarrow$ (open subset of S)

s.t. $f_\beta^{-1} \circ f_\alpha : f_\alpha^{-1}(f_\beta(U_\beta)) \longrightarrow f_\beta^{-1}(f_\alpha(U_\alpha))$ is smooth.

[I.e. defining exactly so the change of coor lemma holds]

Eg.



<u>Thm</u>: S a topological surface. Then there exist a coll (f_α, U_α) making it into a smooth surface. Any two such smoothings are diffeomorphic. [Thus class of surfaces doesn't change.]

[<u>Pf</u>: For existence, use a triangulation and do the gluing in a controlled way.]

<u>Note</u>: False in higher dimensions: $S^7 = \{x \in \mathbb{R}^8 \mid |x| = 1\}$ [Query] has 28 non diffeomorphic smoothings! Discuss / poss meanings

———— o ————

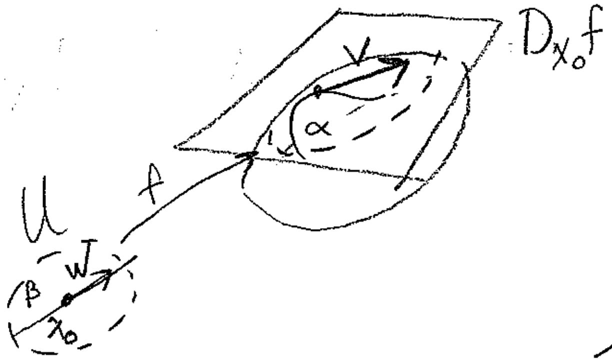
Suppose $c: (-\varepsilon, \varepsilon) \to S \subseteq \mathbb{R}^3$ is a smooth curve w/ $c(0) = p$. Then $c'(0) \in \mathbb{R}^3$ is called a <u>tangent vector</u> to S at p. The collection of all such <u>tangent</u> vectors is the <u>tangent space</u> $T_p S$.

<u>Lemma</u>: Suppose $f: U \to S$ is a coor patch w/ $f(x_0) = 0$

Then $T_p M = \text{image}(D_{x_0} f)$ ← always 2 dim'l.

[In particular, $T_p M$ is a 2 dimensional linear subspace of $\mathbb{R}^3$]

Pf: $(\subseteq)$ Suppose $V = D_{x_0}f(W)$. Let $\beta(t) = x_0 + tW$

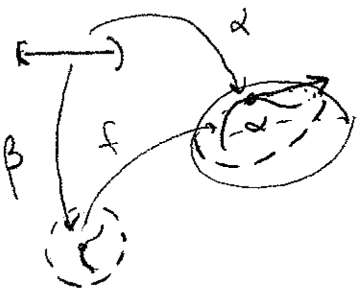


Let $\alpha = f \circ \beta : (-\varepsilon, \varepsilon) \to S$.

Then $\alpha'(0) = D_{x_0}f(\beta'(0))$
$= V$

by chain rule.

$(\supseteq)$ Let α be a curve defining a tangent vector V, shrink ε so that image lies in $f(U)$.



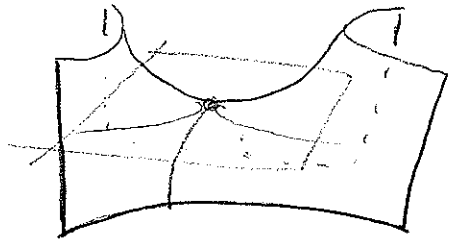
Then $\beta = f^{-1} \circ \alpha$ is smooth and

$\alpha = f \circ \beta$ so

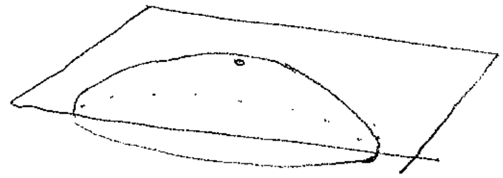
$V = \alpha'(0) = D_{x_0}f(\beta'(0))$ as desired. ∎

Lecture 7: Today: Geometry of curves.

[Quick comment:]



Tangent space = plane that best approximates S at p.



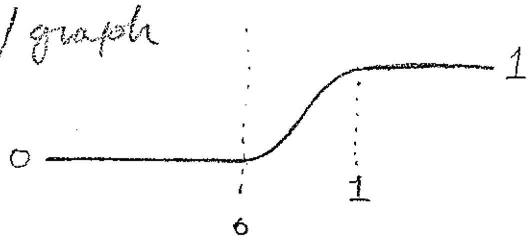
Smooth curve: smooth fn $c : (a,b) \longrightarrow \mathbb{R}^3$.

regular curve: $c'(t) \neq 0$ for all $t \in (a,b)$.

[regular curve is analogous to our smooth surface; only need one chart as the topology of 1 mflds is trivial. Need to avoid.

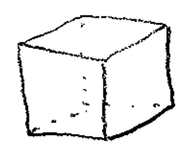
Oddly smooth curve: $c : (-1,1) \to \mathbb{R}^2$ whose image is

$h : \mathbb{R} \to \mathbb{R}$ smooth w/ graph



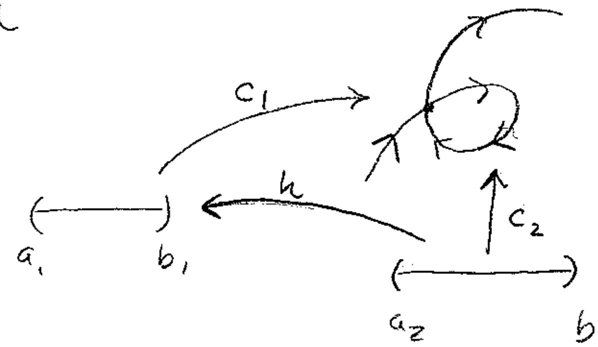
$$h(x) = \begin{cases} 0 & x < 0 \\ \dfrac{f(x)}{f(x) + f(1-x)} & x \in [0,1] \\ 1 & x > 1 \end{cases} \qquad f(x) = e^{-1/x}$$

Take $c(t) = \big(h(t),\ h(t+1)\big)$ to trace out.

N.B.: $\exists$ a smooth map  $\longrightarrow \mathbb{R}^3$ w/ image 

Def: A smooth curve $C_2 : (a_2, b_2) \longrightarrow \mathbb{R}^3$ is a reparameterization of $C_1 : (a_1, b_1) \longrightarrow \mathbb{R}^3$ if $\exists$ a diff $h : (a_2, b_2) \longrightarrow (a_1, b_1)$ s.t. $C_2 = C_1 \circ h$

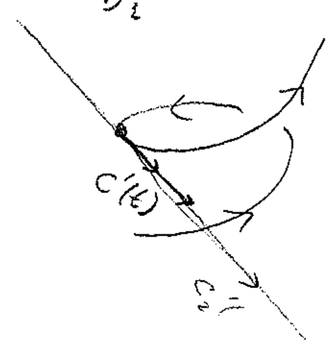
[regard curves as the same if they are reparam.]



[Note: curve not required to be embedded.]

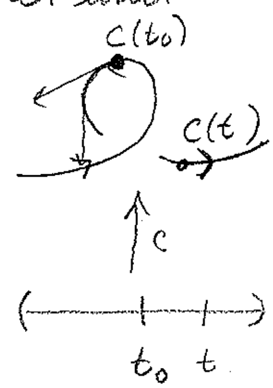
Fix $c : (a, b) \longrightarrow \mathbb{R}^3$ a regular curve

$$T(t) = \frac{c'(t)}{|c'(t)|} \quad \text{unit tangent vector.}$$



[Really what is well defined is the (oriented) tangent line]

Note: Can reparameterize so that c moves at unit speed and $c'(t) = T(t)$.



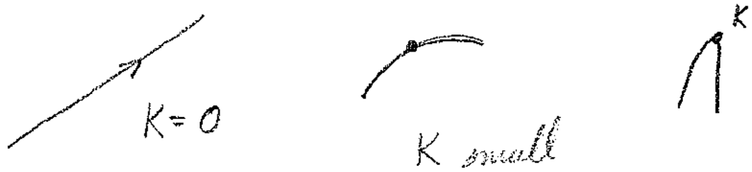
$$\text{dist}(t_0, t_1) \text{ along } c \text{ is } \int_{t_0}^{t_1} |c'(t)|\, dt = d(t)$$

d is smooth, strictly increasing, so gives a diffeo $d : (a, b) \longrightarrow (a', b')$

Then $c_u = c \circ d^{-1}$ is a unit speed param.

For now: let's focus on a unit speed curve.

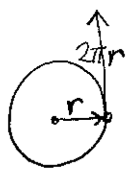
Curvature: quatative measure of how bent the curve is at each point.



Def: C a unit-speed curve, then $K(t) = |T'(t)|$
$= |c''(t)|$

Ex: line has $K=0$

Ex:

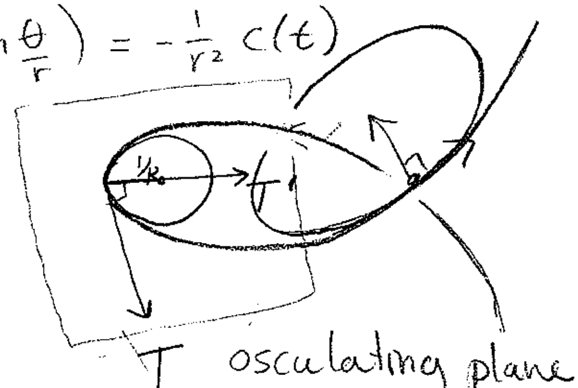


$$c(t) = \left(r\cos\frac{\theta}{r},\ r\sin\frac{\theta}{r}\right)$$

$$c'(t) = \left(-\sin\frac{\theta}{r},\ \cos\frac{\theta}{r}\right)$$ ⟵ unit speed.

$$c''(t) = \left(-\frac{1}{r}\cos\frac{\theta}{r},\ -\frac{1}{r}\sin\frac{\theta}{r}\right) = -\frac{1}{r^2}c(t)$$

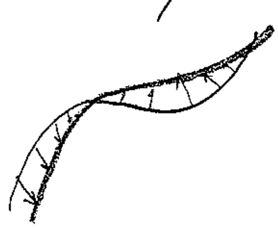
$$K = \frac{1}{r}$$



Geometrically: 1) $\frac{1}{K}$ = turning radius

2) $K(t) = \frac{1}{r}$ where r is the radius of the unique round circle at $c(t)$ whose first two derivatives match w/ c', c''.

3) K measures change in length as we push in the direction of the normal

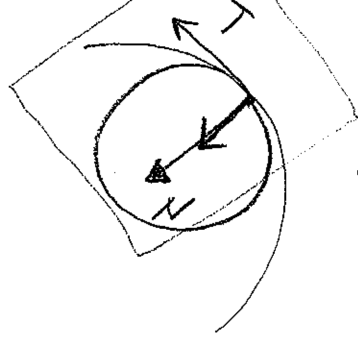


} loosy picture.

Detail: [Accelleration is perp to T as speed is not changing]

$$0 = \frac{d}{dt}\langle T,T\rangle = \langle T',T\rangle + \langle T,T'\rangle = 2\langle T,T'\rangle$$

Set $N(t) = \dfrac{T'}{|T'|}$

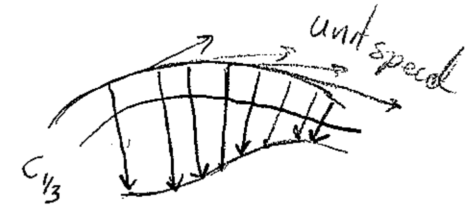


osculating plane

$$\boxed{C'' = K N}$$

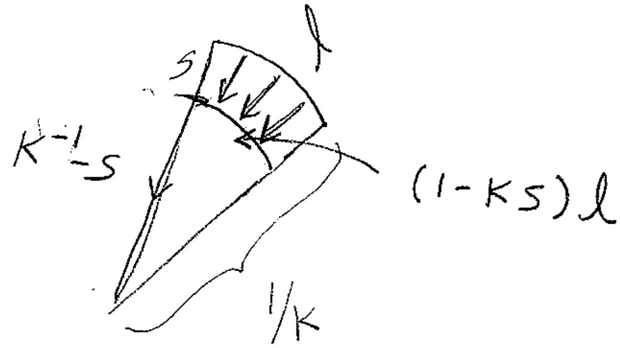
Consider the family of curves $c_s(t) = c(t) + s\,N(t)$

$C : \underset{t}{(a,b)} \times \underset{s}{\mathbb{R}} \longrightarrow \mathbb{R}^3$ a smooth fn.

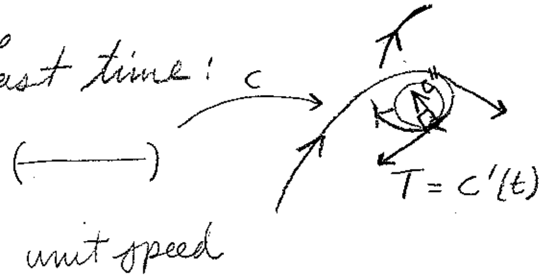


$$L(s) = \int_a^b \left|\frac{\partial c_s(t)}{\partial t}\right| dt \quad \text{length of } c_s$$

$$\left.\frac{dL}{ds}\right|_{t=0} = -\int_a^b K\,dt$$



Lecture 8: Last time:



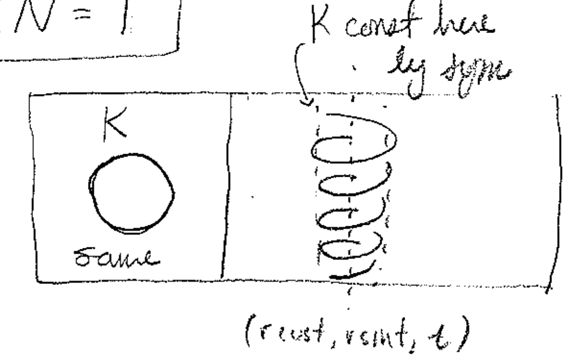
$K(t) = |T'(t)| = |c''(t)|$

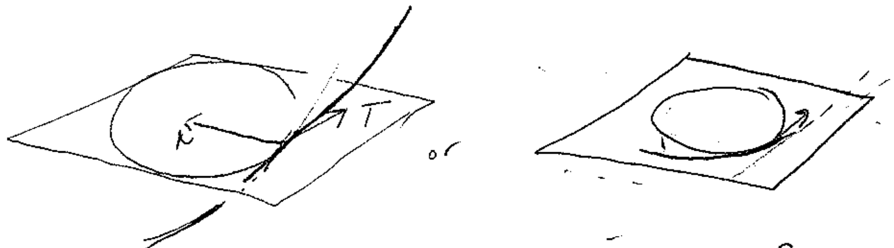
$$N = \frac{c''(t)}{|c''(t)|}$$

$$\boxed{KN = T'}$$

[Curvature measures...]

K captures only part of the info about c.

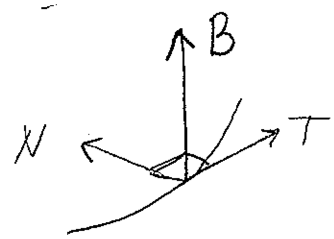


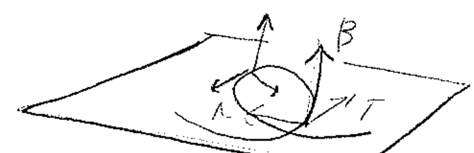


} unmeasured: how c twists w.r.t the osculating plane.

Set $B(t) = T(t) \times N(t)$

[Binormal]





Consider $B'(t)$, and note: $\langle B', B \rangle = 0$ [same reason as last time]

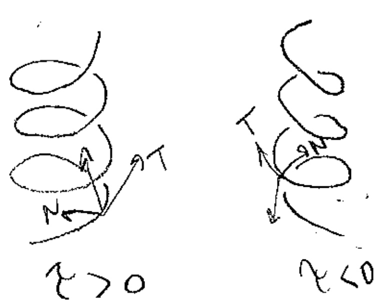
Also $\langle B', T \rangle = 0$ because

$$0 = \frac{d}{dt}\langle B, T \rangle = \langle B', T \rangle + \langle B, T' \rangle = \langle B', T \rangle + \underbrace{\langle B, KN \rangle}_{=0}$$

Hence B' is a scalar mult of N, say

$$B'(t) = -\tau(t)\, N(t)$$

$\tau(t)$ ⟶ torsion of c at t.



Note: $\tau(t)$

c is strongly regular if $K(t) \neq 0$ for all t.

Thm: For a strongly regular unit speed curve,

$$(*) \quad \begin{aligned} T' &= \phantom{-KT + {}} KN \\ N' &= -KT \phantom{{}+KN} + \tau B \\ B' &= \phantom{-KT + {}} -\tau N \end{aligned} \quad \text{for each } t.$$

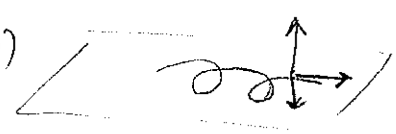
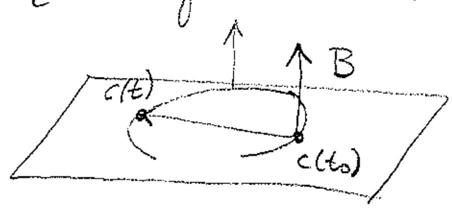
Pf: $\langle N', N \rangle = 0$ for the usual reason.

$$0 = \frac{d}{dt}\langle N, T \rangle = \langle N', T \rangle + \langle N, T' \rangle \Rightarrow \langle N', T \rangle = -K.$$

$$0 = \frac{d}{dt}\langle N, B \rangle = \langle N', B \rangle + \langle N, B' \rangle \Rightarrow \langle N', B \rangle = -\tau.$$ $\blacksquare$

Q: To what extent does K, τ determine c?

Ex: c lies in a plane $\iff \tau = 0$ for all t. $\iff$ B is const

$(\Rightarrow)$  $(\Leftarrow)$ 

$$\frac{d}{dt}\langle c(t) - c(t_0), B(t) \rangle = \langle c'(t), B(t) \rangle + \langle c(t) - c(t_0), B'(t) \rangle = 0.$$

$\Rightarrow \langle c(t) - c(t_0), B(t) \rangle = 0 \quad \forall t$, so lies in a plane. $\blacksquare$

Fundamental Theorem of Curves:

Let $\kappa, \tau : (a,b) \longrightarrow \mathbb{R}$ be smooth fns, w/ $\kappa > 0$.

Then $\exists$ a strongly regular curve $c : (a,b) \longrightarrow \mathbb{R}^3$

w/ curvature and torsion fns equal to κ and τ.

This curve is unique up to translation and rotation.

Idea: $(*)$ are a set of differential equations for T, N, B. For general reasons, they have a solution, say w/ init cond

$$T(t_0) = (1,0,0)$$
$$B(t_0) = (0,1,0)$$
$$N(t_0) = (0,0,1)$$

Then set $c = \int_{t_0}^{t} T(t)\, dt$

Check that $T_c, N_c, B_c = T, N, B$.

and T, N, B are orthonormal.

What about non unit speed curves?

$$T = \frac{c'}{|c'|} \qquad B = \frac{c' \times c''}{|c' \times c''|} \qquad N = B \times T$$

$$\kappa = \frac{|c' \times c''|}{|c'|^3} \qquad \tau = \frac{\langle c' \times c'', c''' \rangle}{|c' \times c''|^2}$$

The reason that you don't solve for N in terms of c'' is that if $c = \underset{\text{unit speed}}{\underline{\underline{c_u}}} \circ g$

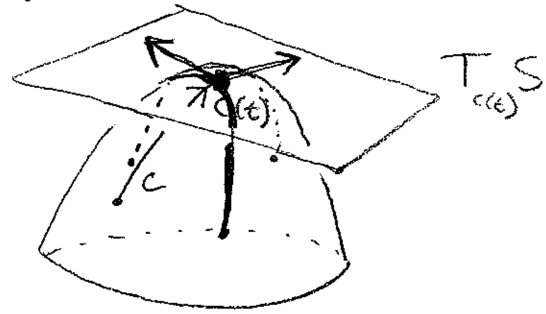
$c'' = \left(c_u'(g(t))\, g(t) \right)' =$

$c_u''(g(t))(g'(t))^2 + c_u'(g(t))\, g''(t)$ don't know.

Lecture 9: Today: Length and area of surfaces.

Length:

$c:(a,b)\longrightarrow S$ curve in surface S.



length of $c = \displaystyle\int_a^b |c'(t)|\,dt \qquad c'(t)\in T_{c(t)}S$

[Can also talk about angles of vectors in $T_{c(t)}S$; both are encoded in this inner product]
[intrinsic geom all comes from this information.]

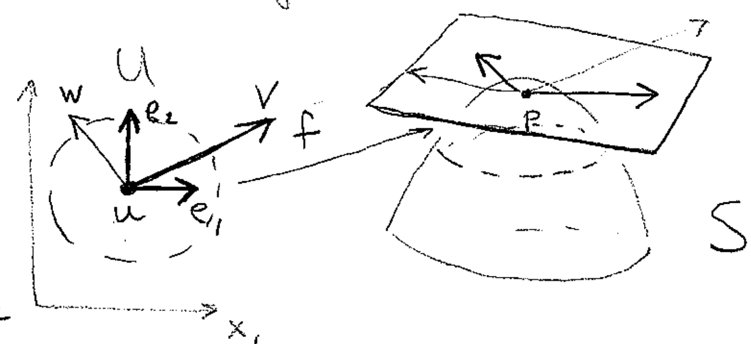
Def: $S\subseteq\mathbb{R}^3$ a smooth surface. The first fundamental form of S at p is the fn $I_p : T_pS\times T_pS\longrightarrow\mathbb{R}$ defined by

$$I_p(v,w)=\langle v,w\rangle$$

[In general, the first fund form is the collection of all such]

I_p is a symmetric bilinear form [Query]

In a coordinate patch:



$e_1=(1,0)$
$e_2=(0,1)$

$$g_{ij}(u)=I_p\Big(\overset{\bar e_1}{\overset{\|}{D_uf(e_i)}},\ \overset{\bar e_2}{\overset{\|}{D_uf(e_j)}}\Big)$$

$$g_{ii}=\text{length}\,(D_uf(e_i))^2$$

$V=(v_1,v_2)=v_1e_1+v_2e_2$
$W=(w_1,w_2)=w_1e_1+w_2e_2$

$$I_p(v_1\bar e_1+v_2\bar e_2,\ w_1\bar e_1+w_2\bar e_2)$$

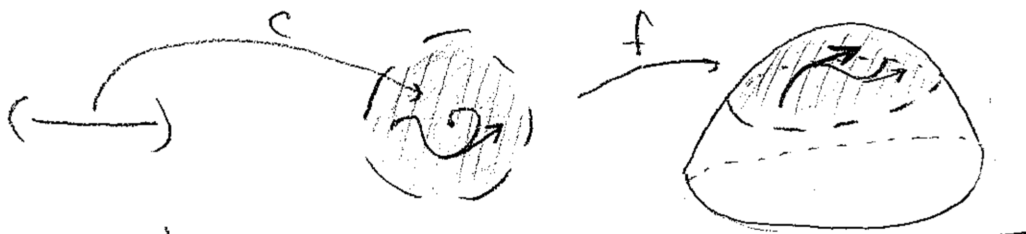
$$I_p\big(D_uf(v),D_uf(w)\big) = v_1w_1g_{11}+v_1w_2g_{12}+v_2w_1g_{21}+v_2w_2g_{12}.$$

$$D_uf(v_1e_1+v_2e_2)=v_1\bar e_1+v_2\bar e_2$$

$$= V \underbrace{\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}}_{\text{metric coeffs } G} W^T$$

Note: $g_{21} = g_{12}$ as I_p is symmetric.

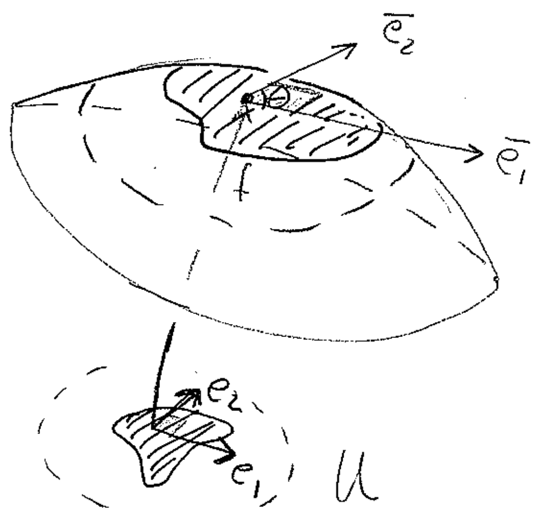
Ex: Suppose c is a curve in S, with image in a patch $f: U \to S$.



$$\text{Length}(f \circ c) = \int_a^b |(f\circ c)'(t)|\, dt = \int_a^b \sqrt{I_p\big(\underbrace{(f\circ c)'(t)}_{(D_{c(t)}f)(c'(t))}, (f\circ c)'(t)\big)}\, dt$$

$$= \int_a^b \sqrt{c'(t)\ G\ c'(t)^T}\ dt.$$

Area: For $A \subseteq f(U) \subseteq S$, the area of A is



$$\int_{f^{-1}(A)} |\bar{e}_1 \times \bar{e}_2|\ dx_1\, dx_2$$

$$= \int_{f^{-1}(A)} \sqrt{\det G}\ dx_1\, dx_2 \quad \text{(should it exist)}$$

as $\det G = g_{11} g_{22} - g_{12}^2 = |\bar{e}_1|^2 |\bar{e}_2|^2 - \underbrace{\langle \bar{e}_1, \bar{e}_2 \rangle^2}_{\text{''}}$ where $|\bar{e}_1|^2|\bar{e}_2|^2\cos^2\theta$ "

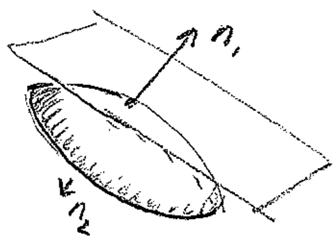
$$= |\bar{e}_1|^2 |\bar{e}_2|^2 (1 - \cos^2\theta) = |\bar{e}_1 \times \bar{e}_2|$$

Lemma: Area does not depend on which chart you use.

Note: Larger sets can be broken into pieces lieing in charts inorder to compute the area.

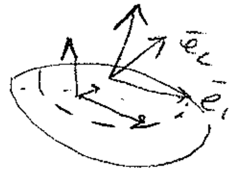
---

normal vectors:



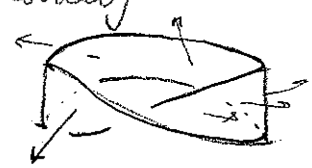
a normal vector to S at p is one $\perp$ to T_pS. [Usually look at unit unit normal vectors]

Over a coor patch can make the choice consistently:



$$n = \frac{\bar{e}_1 \times \bar{e}_2}{|\bar{e}_1 \times \bar{e}_2|}$$

may or may not be able to do so globally.

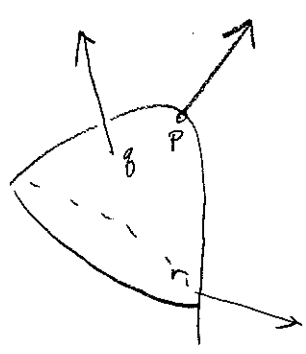


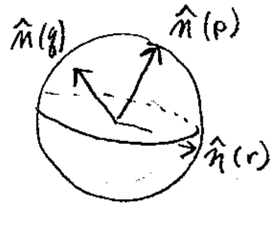
Gauss Map: [tool to define curvature.]

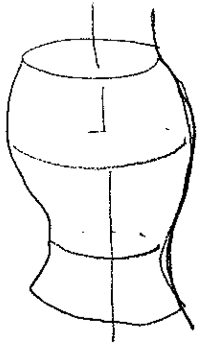
S a surface w/ a consistent choice of unit normal at each pt.

Then set $\hat{n} : S \longrightarrow S^2 = \{ x \in \mathbb{R}^3 \mid |x| = 1 \}$

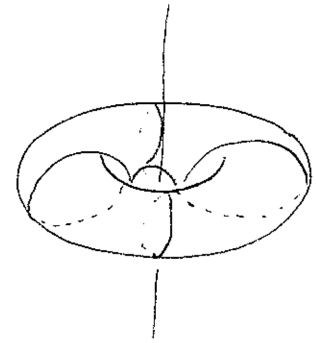
via $\quad p \longmapsto$ unit normal at p.





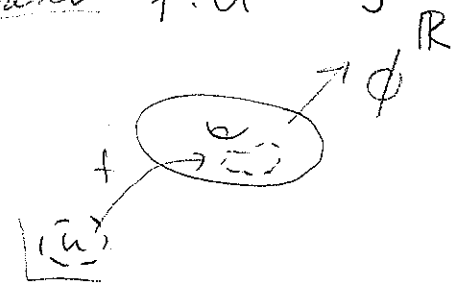
Ex:  surface of revolution

[curve spun must have nowhere 0 velocity vector.]



What does it mean for $\phi: S \to \mathbb{R}$ to be smooth? or $\phi: S_1 \to S_2$?

<u>Def</u>: $\phi: S \to \mathbb{R}$ is smooth if for every <u>coordinate patch</u> $f: U \to S$ we have $\phi \circ f: U \to \mathbb{R}$ is smooth.



<u>Ex</u>: If W is an open set $\supseteq S$, $\phi: W \to \mathbb{R}$ is smooth, then $\phi: S \to \mathbb{R}$ is also smooth.

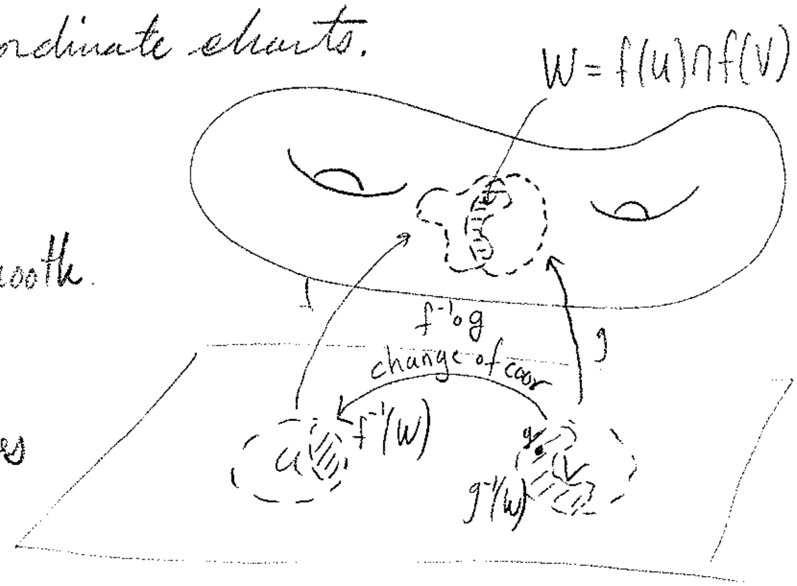
<u>Note</u>: It suffices to check this cond for a collection of coordinate charts that cover S because of.

<u>Change of Coordinates Lemma</u>: $S \subseteq \mathbb{R}^3$ a surface.

Let $f: U \to S$, $g: V \to S$ coordinate charts.

Set $W = f(U) \cap f(V)$. Then

$f^{-1} \circ g: g^{-1}(W) \to f(W)$ is smooth.



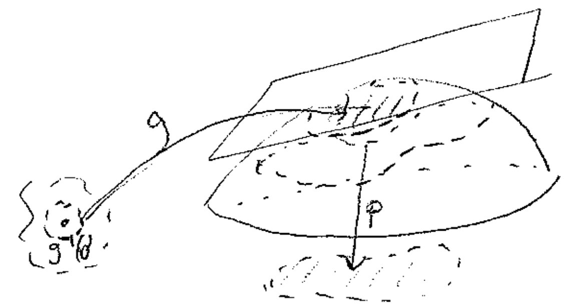
Is $\phi \circ g$ diff at $y \in f^{-1}(W)$? Yes as

$$\phi \circ g = (\phi \circ f) \circ (f^{-1} \circ g)$$

<u>Lemma</u>: Let S be a smooth surface in $\mathbb{R}^3$, given $p \in S$, we can permute the coordinates so that there is a coor patch $f: \mathcal{U} \to S$ w/ $f(\mathcal{U}) \ni p$ of the form $f(x,y) = (x, y, h(x))$.

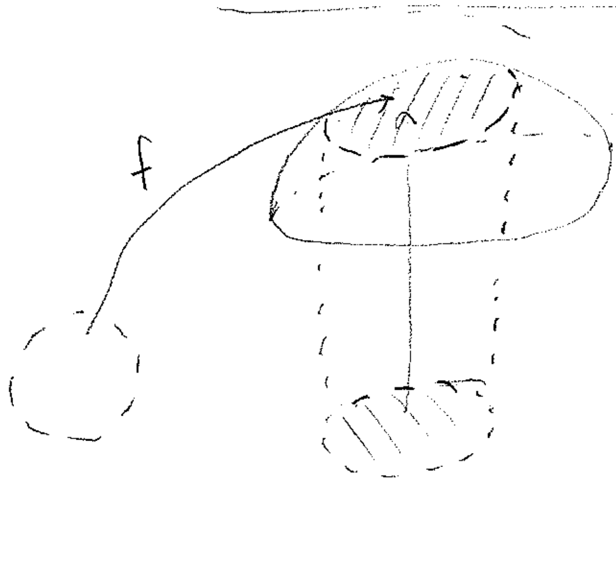
Pf: Take some coor patch $g: \mathcal{U} \to S$ containing p. Let $T = \text{image}(D_{g^{-1}(p)}\, g)$

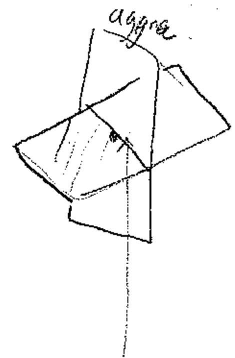
By permuting the vars, can assume the proj $P: \mathbb{R}^3 \to \mathbb{R}^2$, $(x,y,z) \mapsto (x,y)$ is surjective when restricted to T.



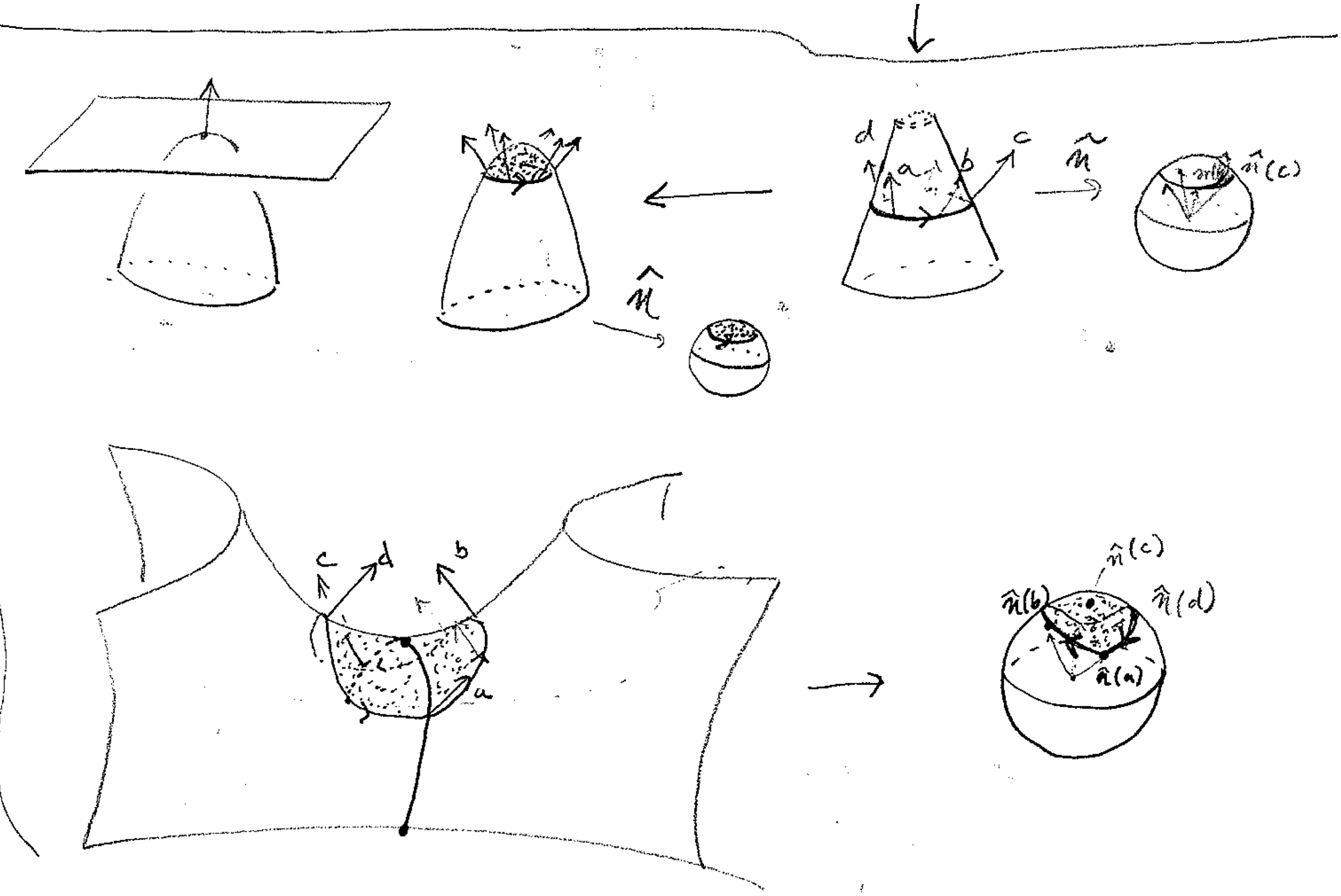
Then by the inverse function thm, $P \circ g$ is a diffeo when restricted to some nbhd W of $g^{-1}(p)$. Take $f = g \circ (P \circ g)^{-1}$. ∎

Why does it imply the change of coor lemma?





Ex:



didn't get to.

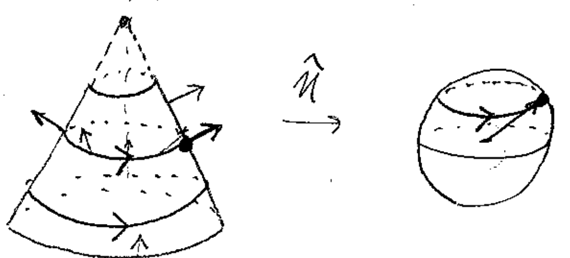
Lecture 10: Today: Curvature 101.

Last time: $S \subseteq \mathbb{R}^3$ smooth surface, with consistent unit normal

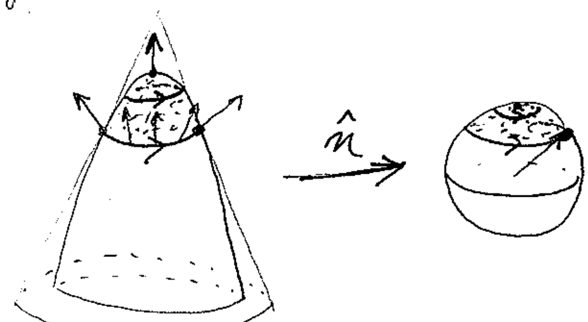
Gauss map: $\hat{n} : S \longrightarrow S^2 = \{ x \in \mathbb{R}^3 \mid |x| = 1 \}$

$p \longmapsto$ unit normal at p.

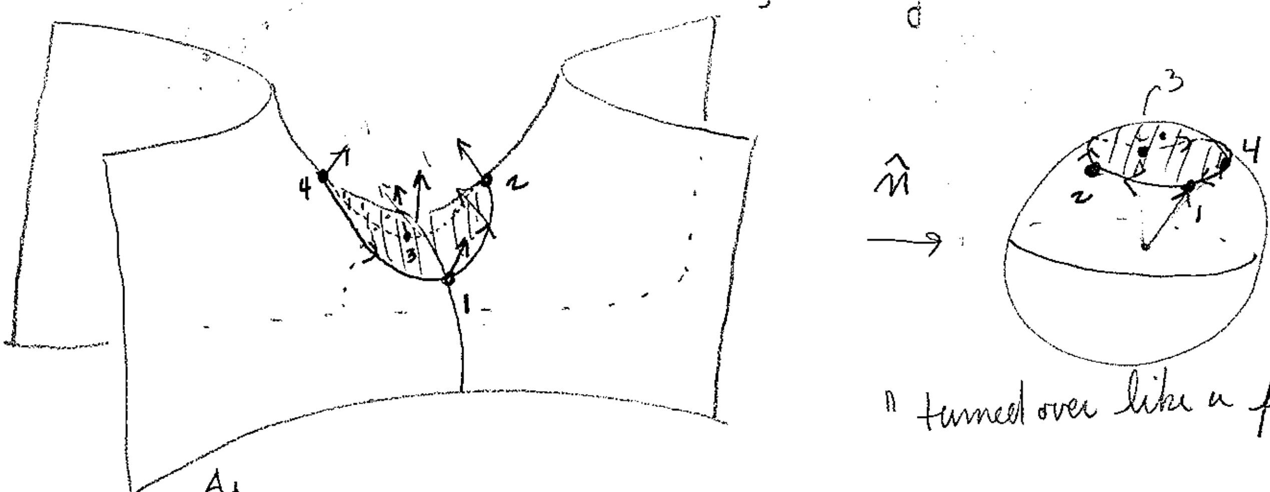
[Examples from last time, plane, cylinder, sphere, ...]



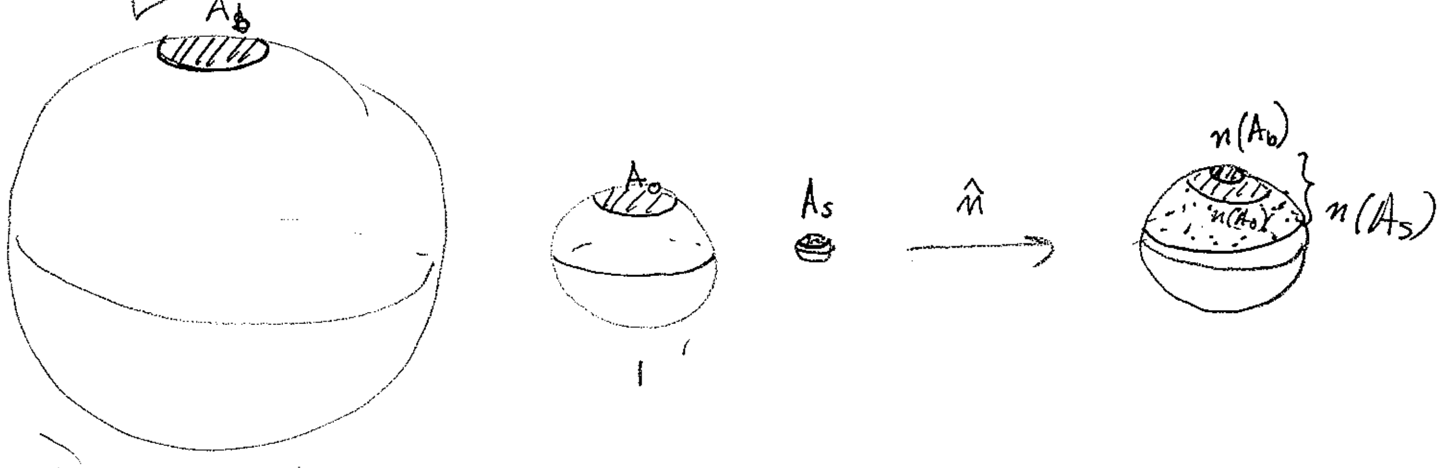
what is the image here.



tangent spaces agree along



"turned over like a pancake"

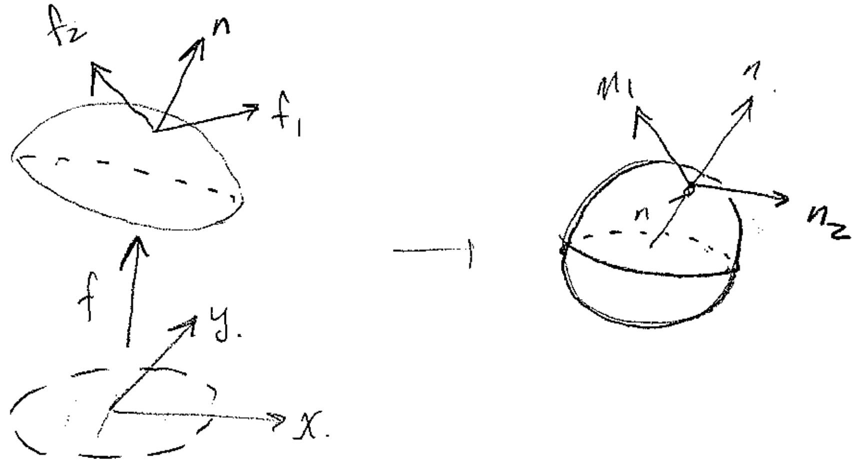


Gauss curvature: $$K(p) = \lim_{A \to \{p\}} \frac{\mathrm{Area}_0(\hat{n}(A))}{\mathrm{Area}(A)}$$

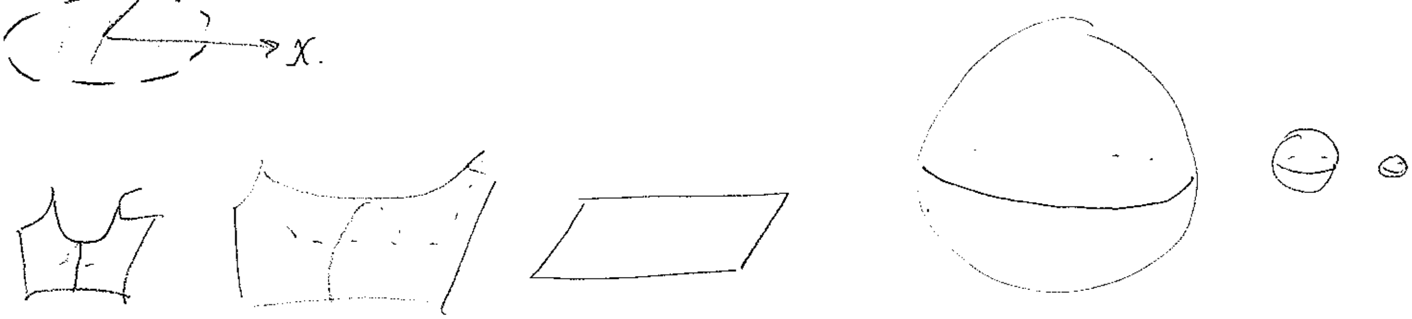
where $A \subseteq f(U)$ a coordinate patch

$$\mathrm{Area}(A) = \int_{f^{-1}(A)} \langle f_1 \times f_2, n \rangle \, dx\, dy \qquad f_1 = \frac{\partial f}{\partial x} = \bar{e_1} \qquad f_2 = \frac{\partial f}{\partial y} = \bar{e_2}$$

$$\mathrm{Area}_0(A) \int_{f^{-1}(A)} \langle n_1 \times n_2, n \rangle \, dx\, dy \qquad n_1 = \frac{\partial n}{\partial x} \qquad n_2 = \frac{\partial n}{\partial y}.$$



$$D_p \hat{n}(f_i) = n_i$$



$K = 0$

$-\infty.$ $\qquad +\infty$

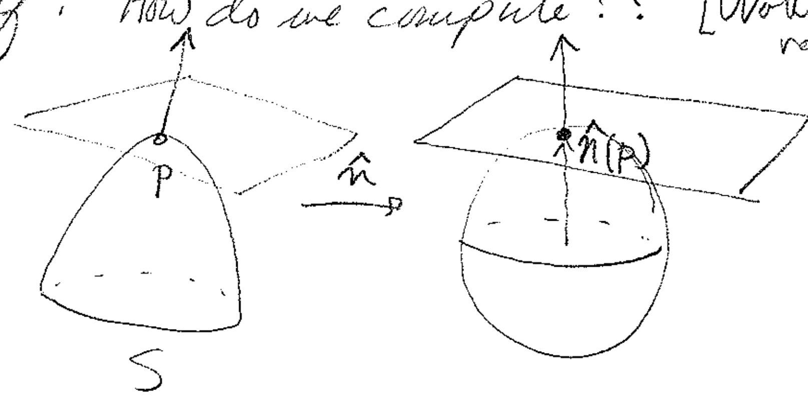
Dilating the surface $S \longrightarrow rS$ changes $K(rS, rp)$

$$= \frac{1}{r^2} K(S, p).$$

Problem: Is this well def? How do we compute?? [Note intr. nature]

Alternate approach:

Weingarten map:

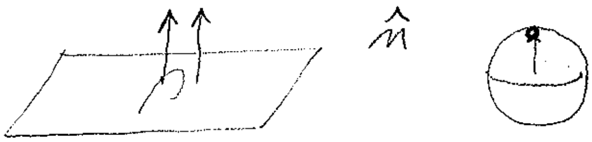


$$D_p\hat{n} : T_pS \to T_{\hat{n}(p)}S^2$$

Note: These are the same plane! If we identify them, get

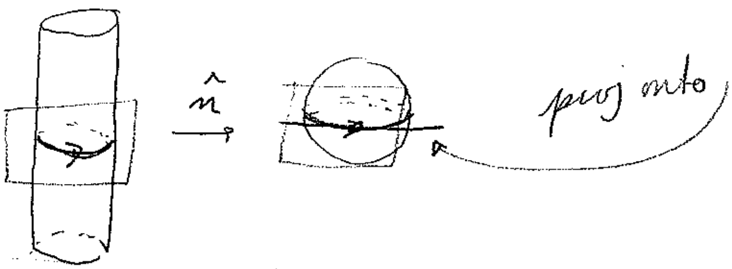
$$L : T_pS \to T_pS \quad \text{a linear map.}$$

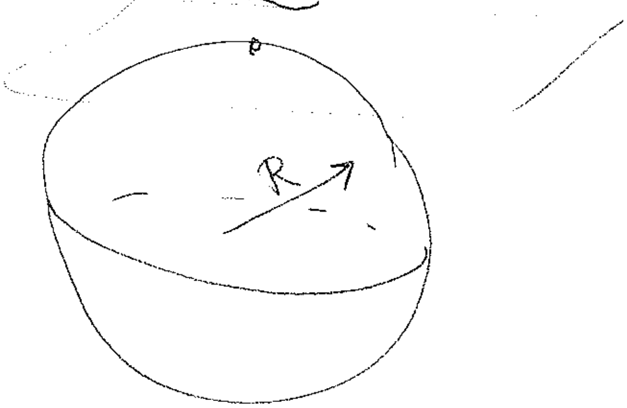
Ex:



$L = 0$

Note extrinsic nature.





$$L = \frac{1}{R} I.$$

S^2_R

[Why did I emph range of L is the same as the domain]

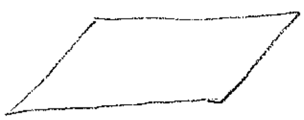
$L: V \longrightarrow W$ doesn't have many invariants.

$L: V \longrightarrow V$ has $\det V$ and $\operatorname{tr} V$

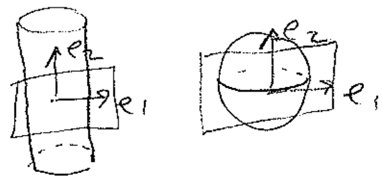
<u>Def</u>: Gaussian curvature at p, $K(p) = \det L$

Mean curvature at p, $H(p) = \frac{1}{2} \operatorname{tr} L$

[explain why this makes sense rel our earlier discussion]

<u>Ex</u>:  $L = 0$ $K = 0$, $H = 0$ differ!

intrinsic extrinsic



$L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $K = 0$ $H = \frac{1}{2}$

$K = \frac{1}{R^2}$ $H = \frac{2}{R}$

What is mean curve good for?

Also did isometries.

<u>Def</u>: A surface is <u>minimal</u> if $H(p) = 0$ $\forall p$.

<u>Ex</u>: Soap bubble surface.

<u>Existence</u>: Plateau's problem / Jesse Douglas, Tibor Rado 1930

Lecture 11: Last time: $\hat{n}: S \longrightarrow S^2$ Gauss map

Midterm handed out on Wed. Open notes, book.

$$L = D_p \hat{n} : T_pS \longrightarrow T_pS \quad \text{Weingarten map.}$$

Gaussian curvature: $K(p) = \det L$ [diff. def of area]

Mean curvature: $H(p) = \frac{1}{2} \operatorname{tr} L$

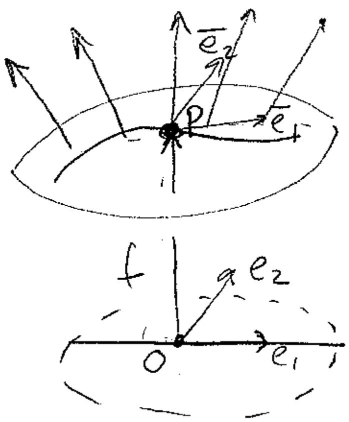
Today: more about L, and how K, H relate to curvature of curves.

Lemma: L is self-adjoint, i.e. $\langle Lv, w \rangle = \langle v, Lw \rangle$

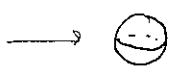
Equiv, the matrix of L w.r.t. an orthonormal basis is symmetric.

[e_1, e_2 then $L_{ij} = \langle Le_j, e_i \rangle$]

Pf: Let $f: U \to S$ be a chart w/ $f(0) = p$. May assume v and w are linearly indep and



$$v = \bar{e}_1 = D_0 f(e_1) = \frac{\partial f}{\partial x} \qquad w = \bar{e}_2$$



Set $N = \hat{n} \circ f : U \to S^2$

Note $L(v) = D_p \hat{n}(v) = D_0 N(e_1) = \dfrac{\partial N}{\partial x}$

$$L(w) = \frac{\partial N}{\partial y} \qquad =$$

Note: $\left\langle \dfrac{\partial f}{\partial x}, N \right\rangle \left[= 0\right]$ on U. (Query) $\Longrightarrow \left\langle \dfrac{\partial f}{\partial y \partial x}, N \right\rangle + \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial N}{\partial y} \right\rangle = 0$

$\left\langle \dfrac{\partial f}{\partial y}, N \right\rangle = 0 \Longrightarrow \left\langle \dfrac{\partial f}{\partial x \partial y}, N \right\rangle + \left\langle \dfrac{\partial f}{\partial y}, \dfrac{\partial N}{\partial x} \right\rangle = 0$

$\Longrightarrow$ at p $\quad \langle v, L(w) \rangle = \langle w, L(v) \rangle$. ∎

Def: The 2nd fundamental form at p is def by

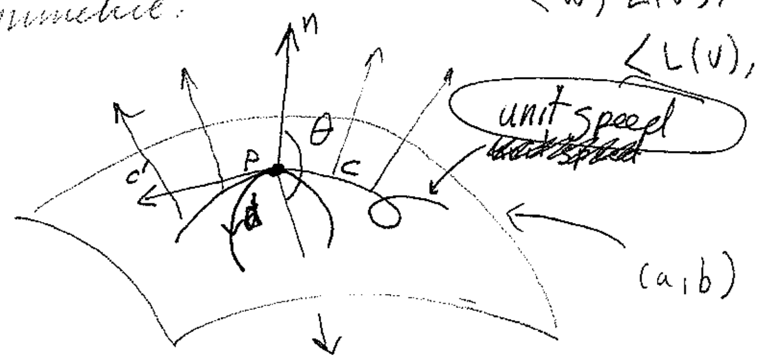
$$\mathrm{II}_p : T_pS \longrightarrow T_pS$$
$$(v,w) \longmapsto \langle L(v), w\rangle$$

$$\mathrm{II}_p(v, w_1+w_2)$$

$$\mathrm{II}_p(w,v) = \langle L(w), v\rangle = \langle w, L(v)\rangle = \langle L(v), w\rangle$$

Note: II_p is bilinear and symmetric:

What does II_p measure??



normal curvature of c at p:

$$K_n = -K\,\langle N_c, n\rangle = -K\cos\theta.$$

N_c – curve normal

K – curvature of c at p [Measures external curvature.]

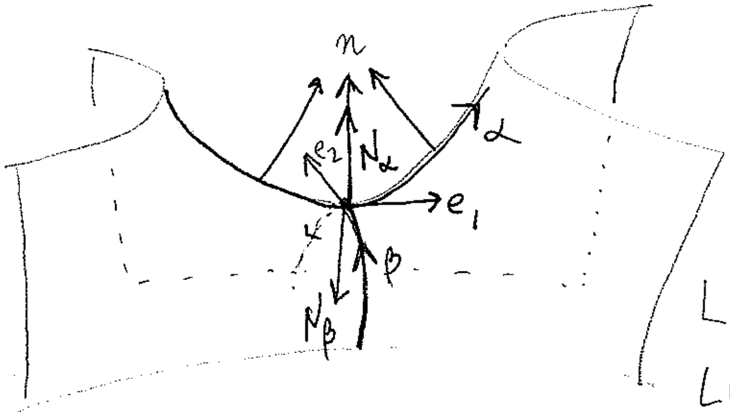
Thm: $\mathrm{II}_p(c', c') = K_n$. [Note: only depends on c'!]

Pf: Assume c is unit speed. Let $n(t)$ be the normal to S at $c(t)$

As $\langle n(t), c'(t)\rangle = 0$ we have

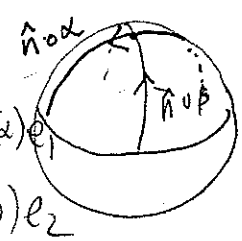
$$\langle n'(t), c'(t)\rangle = -\langle n(t), c''(t)\rangle = -\langle n(t), K(t)N_c(t)\rangle = K_n$$

$$\langle n'(t), c'(t)\rangle = \langle D_p\hat{n}(c'(t)), c'(t)\rangle = \mathrm{II}_p(c', c') \qquad \blacksquare$$



$$\mathrm{II}_p(e_1,e_1) = -K(\alpha)$$

$$\mathrm{II}_p(e_2,e_2) = -K(\beta)$$



$$L(e_1) = -K(\alpha)e_1$$
$$L(e_2) = K(\beta)e_2$$

$$L = \begin{pmatrix} -K(\alpha) & 0 \\ 0 & -K(\beta)\end{pmatrix}$$

Hence: $$K(p) = -K(\alpha)K(\beta)$$
$$H(p) = \frac{1}{2}\left(-K(\alpha) + K(\beta)\right)$$

In general as L is symmetric, $\exists$ an orthonormal basis e_1, e_2 of T_pS so that L is diagonal $L = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$ $\underbrace{K_1 \geq K_2}_{\text{principal curvatures.}}$

For any unit vector $v = \sin\theta\, e_1 + \cos\theta\, e_2$

we have
$$\mathrm{II}_p(v,v) = \langle L(v), v\rangle = \langle K_1 \sin\theta\, e_1 + K_2 \cos\theta\, e_2, v\rangle$$
$$= K_1 \sin^2\theta + K_2 \cos^2\theta$$

Geometrically:

K_1 = max normal curve over all curves $c \subseteq S$ passing through p.

K_2 = min ...

Note: L is only defined up to sign, so is H, K_1, K_2 Because there are two choices for n,

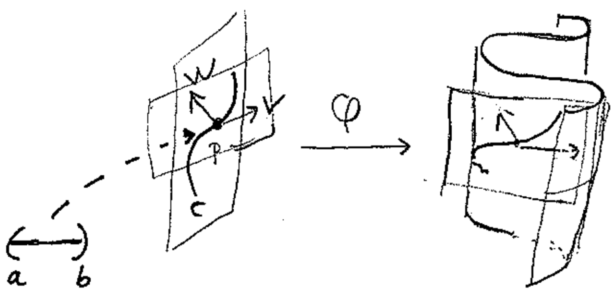
K is well defined, regardless.

Lecture 12: Intrinsic vs. Extrinsic.

Def: $\varphi : S_1 \to S_2$ be a smooth map of surfaces in $\mathbb{R}^3$.
Then φ is a <u>local isometry</u> if $\forall\, p$ and $v, w \in T_p S_1$

we have

$$I_{S_1, p}(v, w) = I_{S_2, \varphi(p)}\left(D_p\varphi(v),\, D_p\varphi(w) \right).$$



φ is an <u>isometry</u> if it is <u>also</u> a diffeomorphism.

Def: A property is <u>intrinsic</u> if it is invariant under isometries.

<u>Intrinsic</u>: • Length of a curve

$$\operatorname{len}(c) = \operatorname{len}(\varphi \circ c)$$

$$\| \qquad\qquad \|$$

$$\int_a^b \sqrt{I_{c(t)}(c', c')}\, dt \qquad \int_a^b \sqrt{I_{c\circ\varphi(t)}\big((\varphi\circ c)'(t),\ \text{same}\big)}\, dt$$

$$(\varphi\circ c)'(t) = D_{c(t)}\varphi(c'(t))$$

• area [come back to this]

<u>Extrinsic</u>: • dist between points in $\mathbb{R}^3$.

• mean curvature.

<u>Notes</u>: • local isometries are local diffeomorphisms [Query?]

• φ is a local isom if $\forall$ charts $U \to S_1$

we have

$$\underbrace{g_{ij}^{S_1}}_{\text{metric coeffs for } I_{S_1}} = g_{ij}^{S_2} \Big\}\ \text{metric coeffs for } U \xrightarrow{\varphi\circ f} S_2$$

[relate back to area.]

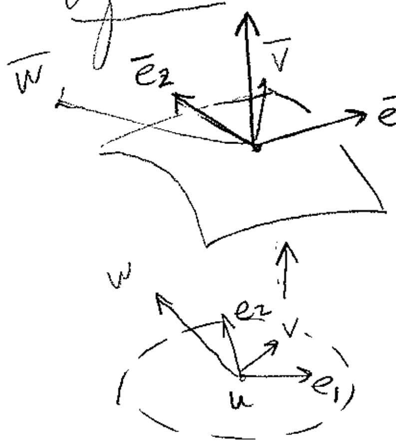
Theorema Egregium ("remarkable theorem")

$\varphi : S_1 \to S_2$ is a local isometry. Then $\forall\, p \in S_1$, we have

$$K_{S_1}(p) = K_{S_2}(\varphi(p))$$

[i.e. Gaussian curvature is <u>intrinsic</u>.] [Explain why it is surprising]

Pf idea: express K in terms of g_{ij} and its derivatives.



$$\bar{e}_1 = Df(e_1) = \frac{\partial f}{\partial x} = f_x \qquad \bar{e}_2 = f_y$$

$$g_{ij}(u) = I_p(\bar{e}_i, \bar{e}_j) \qquad g_{12} = g_{21}$$

$$l_{ij}(u) = II_p(\bar{e}_i, \bar{e}_j) \qquad l_{12} = l_{21}$$

$$(L_{ij}) = \text{matrix of Weingarten map w.r.t. } \{\bar{e}_1, \bar{e}_2\}$$

[Focus on this as it is symmetric, whereas typically isn't.]

v, w (column vectors) in terms of e_1, e_2 $\quad I_p(\bar{v}, \bar{w}) = v^T (g_{ij}) w$

$$II_p(\bar{v}, \bar{w}) = v^T (l_{ij}) w$$

$$= ((L_{ij}) v)^T (g_{ij}) w$$

$$= v^T (L_{ij})^T g_{ij}\, w \implies (l_{ij}) = (L_{ij})^T g_{ij}$$

taking transpose

$$\implies (l_{ij}) = (g_{ij})(L_{ij}) \implies (L_{ij}) = (g_{ij})^{-1}(l_{ij})$$

$$\implies K = \det(L_{ij}) = \frac{\det(l_{ij})}{\det(g_{ij})} = \frac{l_{11} l_{22} - l_{12}^2}{g_{11} g_{22} - g_{12}^2}$$

need can be expressed in terms of g_{ij}

$$\begin{pmatrix} l_{12} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}$$

Take the normal $n = \dfrac{\bar{e}_1 \times \bar{e}_2}{|\bar{e}_1 \times \bar{e}_2|} : U \longrightarrow S^2$

$= \hat{n} \circ f$ basis for $\mathbb{R}^3 = (f_x, f_y, n)$

$$f_{xx} = \Gamma_{11}^1 f_x + \Gamma_{11}^2 f_y - l_{11} n$$

lemma: $\langle f_x, n \rangle = 0 \Rightarrow$

$\langle f_{xx}, n \rangle = -\langle f_x, n_x \rangle$, $n_x = L(f_x)$

$$f_{xy} = \Gamma_{12}^1 f_x + \Gamma_{12}^2 f_y - l_{12} n$$

$$f_{yy} = \underbrace{\Gamma_{22}^1 f_x + \Gamma_{22}^2 f_y}_{\text{definition: } \Gamma^i_{jk} \text{ - Christoffel symbol}} - l_{22} n$$

$$\frac{1}{2}(g_{11})_x = \langle f_{xx}, f_x \rangle = \Gamma_{11}^1 g_{11} + \Gamma_{11}^2 g_{12}$$

$$(g_{12})_x - \frac{1}{2}(g_{11})_y = \langle f_{xx}, f_y \rangle = \Gamma_{11}^1 g_{12} + \Gamma_{11}^2 g_{22}$$

$\Rightarrow$ have linear system w/ det $g_{11}g_{22} - g_{12}^2 > 0$

$|\bar{e}_1|^2 |\bar{e}_2|^2 - |\langle \bar{e}_1, \bar{e}_2 \rangle|^2$

for Γ_{11}^1, Γ_{11}^2 interms of derivatives of g_{ij}

Same for rest $\Rightarrow \Gamma^i_{jk}$ are determined by g_{ij}.

A calculation then shows:

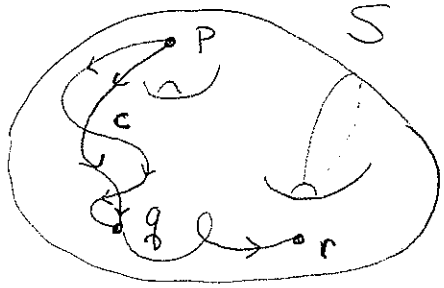
$$l_{11} l_{22} - l_{12}^2 = \sum_{r=1}^{2} g_{1r} \left(\frac{\partial \Gamma^r_{22}}{\partial x} - \frac{\partial \Gamma^r_{21}}{\partial y} + \sum_{m=1}^{2} \left(\Gamma^m_{22} \Gamma^k_{m1} - \Gamma^r_{m1} \Gamma^r_{m2} \right) \right)$$

$\Rightarrow$ Theorema Egregium.

Lecture 13: Last time: Intrinsic v. Extrinsic

Today: Geodesics and distances.

Intrinsic Distance:

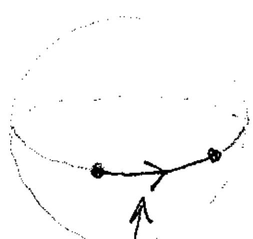


$$d(p,q) = \inf \{ \operatorname{len}(c) \mid c \text{ a path in } S \text{ joining } p \text{ to } q \}$$

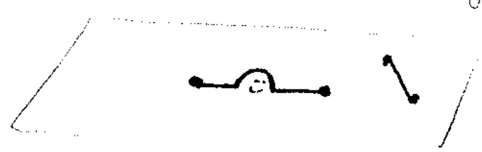
Ex: This makes S into a metric space

Fact: Provided S is closed in $\mathbb{R}^3$, $d(p,q) = \operatorname{len}(c)$ for some particular c.

$S = \mathbb{R}^2 \setminus \{0\}$ $\quad d(p,q) = $ Euc. dist



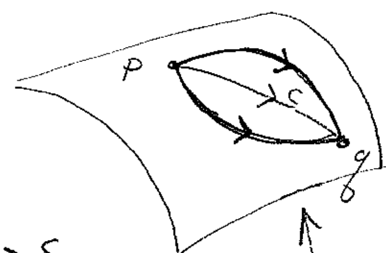
vs.



geodesics are such length minimizing paths.

Variational Characterization:

$\begin{bmatrix}\text{unit}\\ \text{speed}\end{bmatrix}$ $c : [a,b] \longrightarrow S$



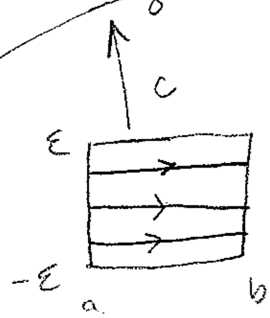
$c_\alpha : [a,b] \longrightarrow S$

$c_\alpha(t) = C(\alpha, t)$

$C : (-\varepsilon,\varepsilon)\times[a,b] \longrightarrow S$

w/ $C(\cdot, a) = p$ $\quad C(\cdot, b) = q$

and $c_0 = c$

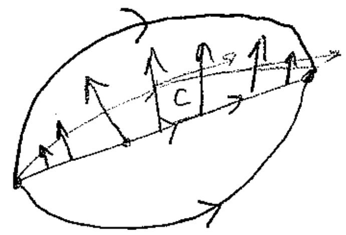


Talk about expectation

If c is a geodesic, then $\operatorname{len}(c) \le \operatorname{len}(c_\alpha)$ for all α.

Hence

$$0 = \left.\frac{d\,\mathrm{len}(c_\alpha)}{d\alpha}\right|_{\alpha=0} = \frac{\partial}{\partial\alpha}\left(\int_a^b \sqrt{\langle \underbrace{c_\alpha'(t)}_{=\frac{\partial c}{\partial t}(t,\alpha)}, c_\alpha'(t)\rangle}\, dt\right)\Bigg|_{\alpha=0}$$



$$= \int_a^b \left(\frac{1/2}{\sqrt{\langle\,,\,\rangle}} \cdot 2\left\langle \frac{\partial c}{\partial\alpha\partial t}(\alpha,t), \frac{\partial c}{\partial t}(\alpha,t)\right\rangle\right)\Bigg|_{\alpha=0} dt$$

$$= \int_a^b \langle V'(t), c'(t)\rangle\, dt$$

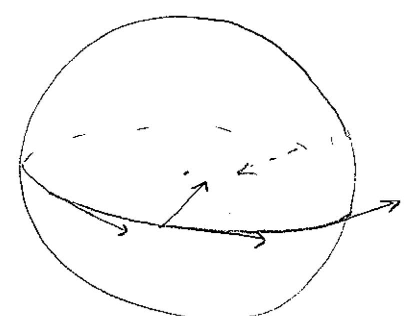
$V(t) = \frac{\partial c}{\partial\alpha}(0,t)$

$$= \underbrace{\langle V(b), c'(b)\rangle}_{=0} - \underbrace{\langle V(a), c'(a)\rangle}_{=0} - \int \langle V(t), c''(t)\rangle dt$$

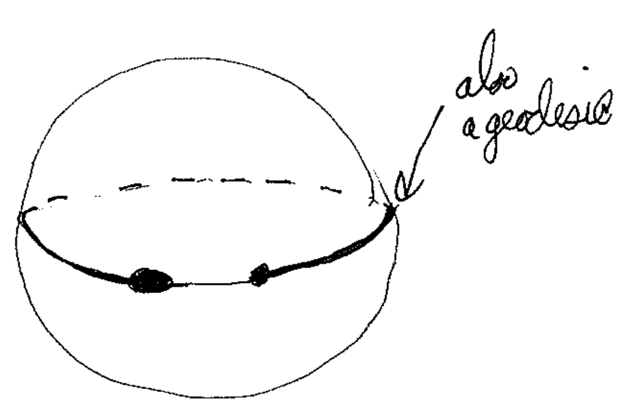
$$= -\int \langle V(t), c''(t)\rangle\, dt$$

So, c a geodesic $\Rightarrow$ $c''(t)$ is <u>normal</u> to S for all t.

[<u>Actually:</u>]



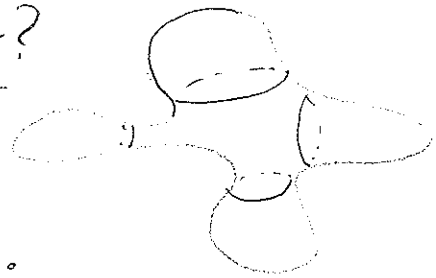
<u>Def</u>: A <u>geodesic</u> in S is a curve $c:(a,b) \to S$ such that $c''(t)$ is normal to S for all t.

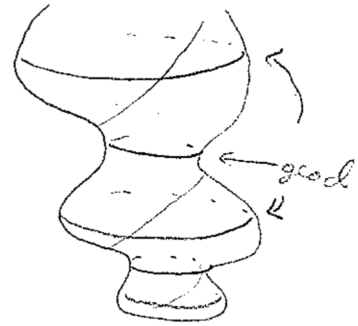


<u>Note</u>, need not minimize length.

<u>Note</u>: is <u>intrinsic</u>, by var. char.

Do geodesics always exist?

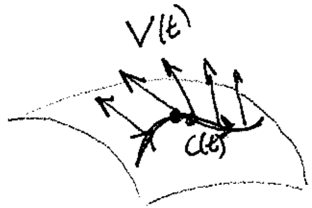




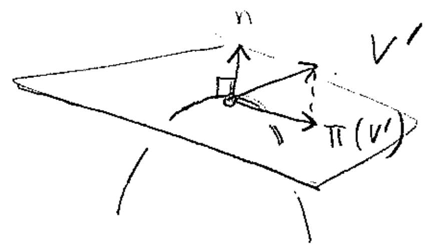
Covariant differentiation:

$c : (a,b) \longrightarrow S$ a curve

$V : (a,b) \longrightarrow \mathbb{R}^3$ a vector field along c, i.e. $V(t) \in T_{c(t)}S$



$$\frac{DV}{dt}(t) = \text{Projection onto } T_{c(t)} \text{ of } V'$$

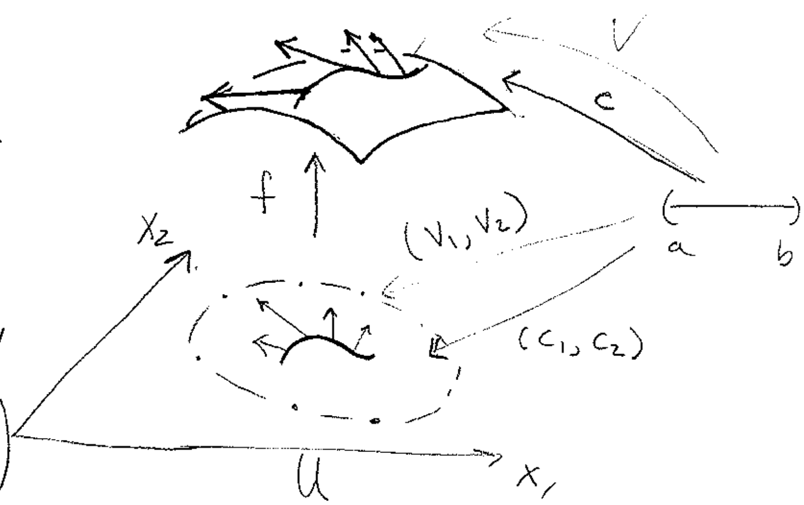


Note: c is a geodesic iff $\frac{Dc'}{dt} = 0$ for all t.

In local coors.

$$V(t) = v_1(t) f_{x_1}(c_1(t), c_2(t)) + v_2(t) f_{x_2}(c_1(t), c_2(t))$$

$$V' = \sum_{i=1}^{2} \left(v_i' f_{x_i} + v_i f_{x_i x_1} c_1' + v_i f_{x_i x_2} c_2' \right)$$

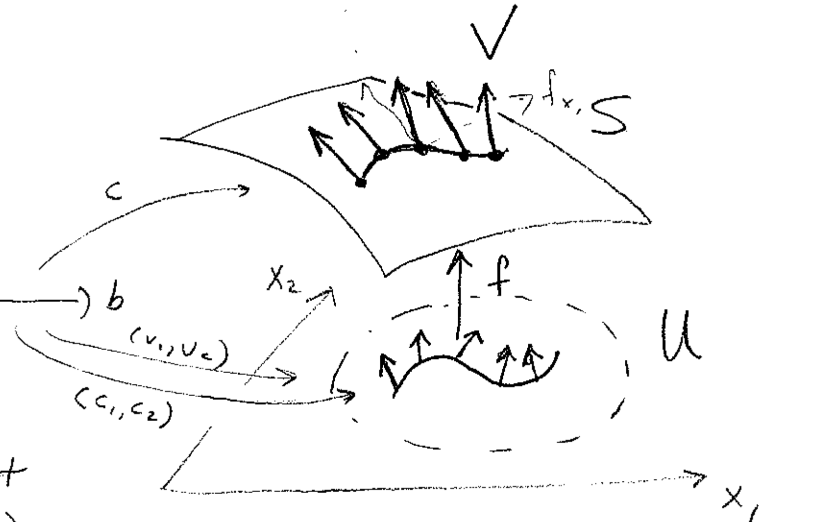


$$f_{x_i x_j} = \Gamma_{ij}^{1} f_{x_1} + \Gamma_{ij}^{2} f_{x_2} - l_{ij} n$$ throw away

$$\frac{DV}{dt} = \sum_{i=1}^{2} \left(v_i' + \sum_{j,k=1}^{2} \Gamma_{jk}^{i} v_j c_k' \right) f_{x_i} \Rightarrow \frac{DV}{dt} \text{ is intrinsic.}$$

Lecture 14: Last time: Copy covariant diff from prev. page.

In local coordinates:

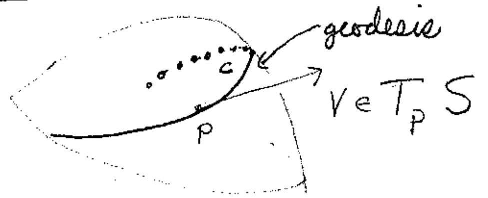


$$V(t) = v_1(t) f_{x_1}(c_1(t), c_2(t)) + v_2(t) f_{x_2}(c_1(t), c_2(t)) \quad a \;(\longrightarrow)\; b$$

$$V' = \sum_{i=1}^{2} \left(v_i' f_{x_i} + v_i f_{x_i x_1} c_1' + v_i f_{x_i x_2} c_2' \right)$$

$$f_{x_i x_j} = \Gamma_{ij}^1 f_{x_1} + \Gamma_{ij}^2 f_{x_2} - \underbrace{l_{ij} n}_{\text{ignore}}$$

$$\frac{DV}{dt} = \sum_{i=1}^{2} f_{x_i} \left(v_i' + \sum_{j,k=1}^{2} v_j \Gamma_{jk}^i c_k' \right) \Rightarrow \boxed{\frac{dV}{dt} \text{ is intrinsic}}$$



Existence of geodesics:

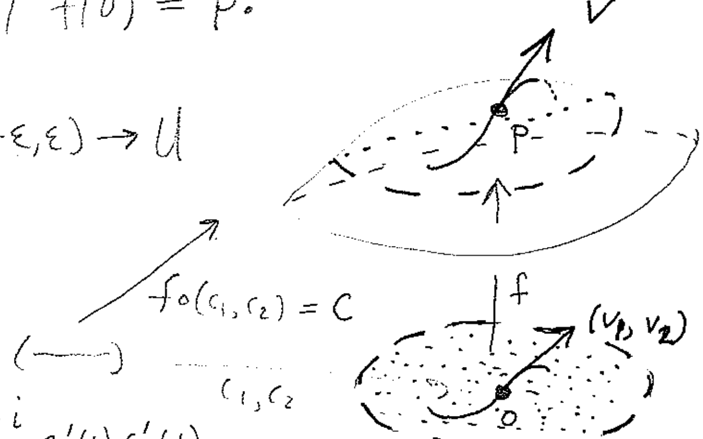
Thm: $S \subseteq \mathbb{R}^3$ a smooth surface. Let $v \in T_p S$. $\exists\ \varepsilon > 0$ and a geodesic $c : (-\varepsilon, \varepsilon) \longrightarrow S$ s.t. $c(0) = p$ and $c'(0) = v$.

Moreover, if $\tilde{c} : (-\delta, \delta) \longrightarrow S$ is another such geod, then $c = \tilde{c}$ on $(-\varepsilon, \varepsilon) \cap (-\delta, \delta)$.

[Note: geodesics are always constant speed.]

Pf: Chose coordinates $f: U \to S$ w/ $f(0) = p$.

Consider a curve in U, $(c_1, c_2): (-\varepsilon, \varepsilon) \to U$



$$c'(t) = c_1'(t) f_{x_1} + c_2'(t) f_{x_2} + ?\, n$$

$$\frac{Dc'}{dt} = 0 \iff c_i'' = -\sum_{j,k=1}^{2} \Gamma_{jk}^i c_j'(t) c_k'(t)$$

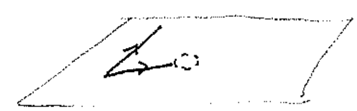
Take $d_i = c_i'$, get a 1^{st} order system

$$d_i = c_i' \ , \ d_i' = -\sum_{j,k=1}^{2} \Gamma_{jk}^i d_j d_k \quad \text{in } (c_i, d_i)$$

By math 2a, there exist a unique solution to these equations with initial cond $c_1(0) = c_2(0) = 0 \quad d_1(0) = v_1 \quad d_2(0) = v_2$

where $v = v_1 f_{x_1} + v_2 f_{x_2}$. ∎

Note: geod may not exist for all time.



Def: A symmetry of S is an isometry $\varphi: S \to S$.

[Skip and put on HW; replace w/ quick discussion of some examples.]

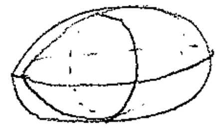
Cor: Suppose φ is a symmetry of S which fixes p in S and $v \in T_pS$. Then the geodesic through p w/ tangent vector v is pointwise fixed by φ.

Pf: Let c be the specified geod. Then $\varphi \circ c$ is also a geod (as geod are intrinsic.)

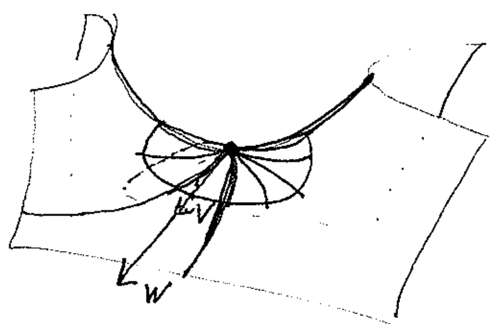
and if $c(0) = p$ then $\varphi \circ c(0) = \varphi(p) = p$
$c'(0) = v$ $\quad (\varphi \circ c)'(0) = (D_p \varphi)\, c'(0) = v.$ $\Rightarrow c = \varphi \circ c$ ∎

<u>Thus</u>: • great circles are the sphere are (all) geod.

• ellipsoid $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$



• surface of revolution



<u>Exponential Map:</u>

$v \in T_p S$. Set

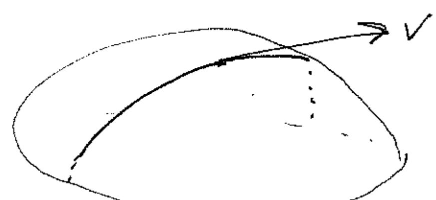
$$\rho_v = \sup\{ r \in \mathbb{R}^+ \mid \exists \text{ a geod } c : (-r, r) \to S \text{ w/ } c(0) = p,\ c'(0) = v \}$$

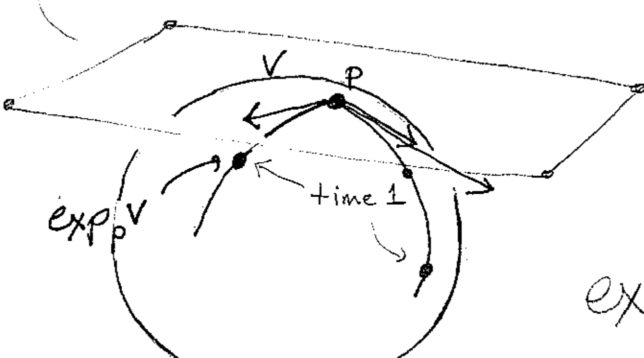
$[\text{pos } \rho_v = \infty]$

move

<u>Note</u>: • $\rho_v > 0$. • $\exists$ a geod $c : (-\rho_v, \rho_v) \to S$ w/ $c'(0) = v$

• $s \in \mathbb{R} \setminus \{0\}$ then $\rho_{sv} = \dfrac{\rho_v}{|s|}$.





$$E_p = \{ v \in T_p M \mid \rho_v > 1 \}$$

$$\exp_p : E_p \longrightarrow S$$

$v \longmapsto c(1)$ where c is the geodesic such that $c(0) = p$ and $c'(0) = v$.

distance traveled equals $|v|$

Note: $\exp_p(s\overset{\text{unit vector}}{V}) : (-\rho_V, \rho_V) \longrightarrow S$ is the geod through p w/ tangent vector V.

Thm: For any p, $\exists$ an open set $0 \in \mathcal{U} \subseteq E_p$ on which $\exp_p$ is smooth.

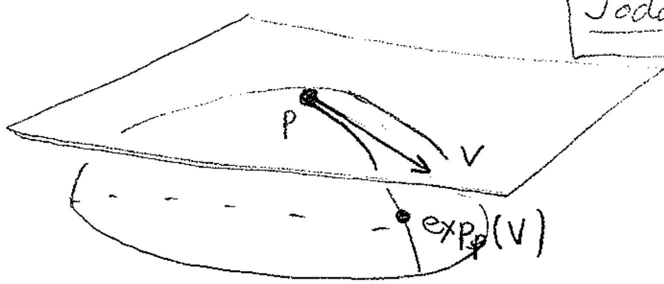
Pf: Math 2b.

Cor: $\exists\, \mathcal{U} \ni 0$ in T_pS such that $\exp_p\big|_{\mathcal{U}}$ is a diffeomorphism.

Pf: $D_0(\exp_p) = \mathrm{Id}$

Lecture 15: Last time: exist of geod and exp. map.

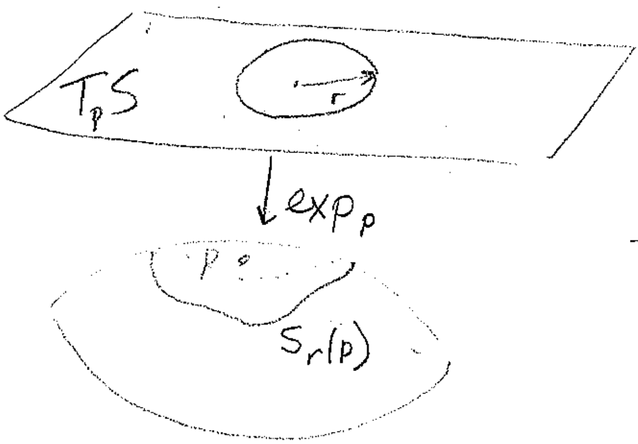
Today: Gauss Lemma and nice local cor.



$$\exp_p : T_pS \longrightarrow S$$
$$v \longmapsto c_v(1)$$

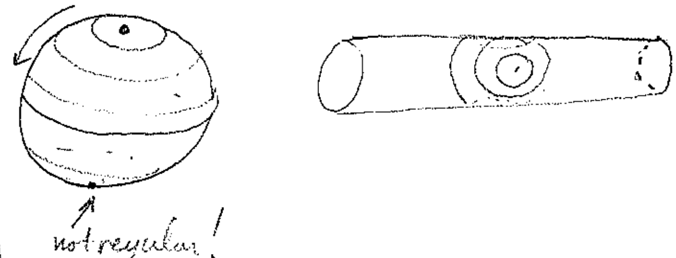
where c_v is the unique geod w/ $c_v(0) = p$, $c_v'(0) = v$

$$S_r(p) = \exp_p\left(\begin{smallmatrix}\text{circle of}\\ \text{radius } r \text{ about } 0\end{smallmatrix}\right) = \left\{x \in S \,\middle|\, \begin{smallmatrix}x \text{ can be joined to } p\\ \text{by a geod of len} = r\end{smallmatrix}\right\}$$



$$B_r(p) = \exp_p\left(\begin{smallmatrix}\text{ball of radius } r\\ \text{about } 0\end{smallmatrix}\right)$$

Note: for small r, $S_r(p)$ is a regular curve, equiv to a circle.



Thm: For small r,

$$S_r(p) = \{q \in S \mid \underbrace{d}_{\text{intrinsic dist}}(p,q) = r\}$$

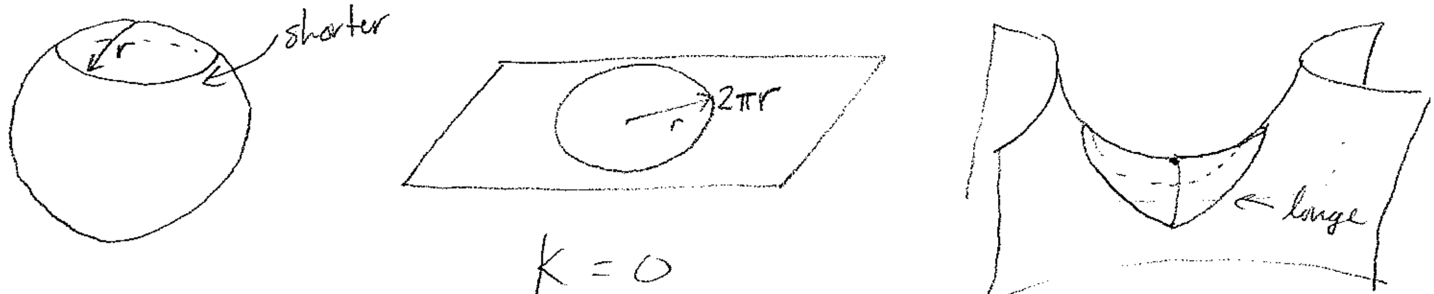
[Will show later.]

$\longleftarrow$ consider skipping

Thm: $\text{Length}(S_r(p)) = 2\pi r\left(1 - \frac{K(p)}{6} r^2 + \text{higher order}\right)$

In particular

$$K(p) = \lim_{r \to 0} \frac{6}{r^2}\left(1 - \frac{L_r}{2\pi r}\right) \implies K(p) \text{ is intrinsic.}$$

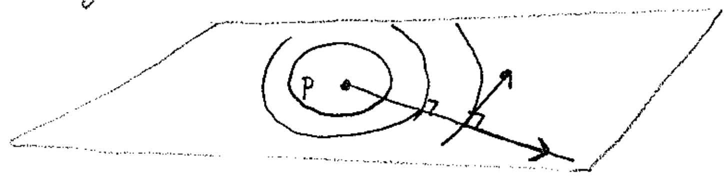


$K > 0$ $\qquad$ $K = 0$ $\qquad$ $K < 0.$

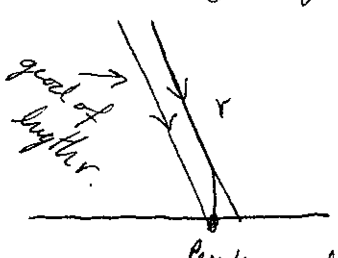
<u>Gauss Lemma</u>: $p \in S$ a smooth surface in $\mathbb{R}^3$.

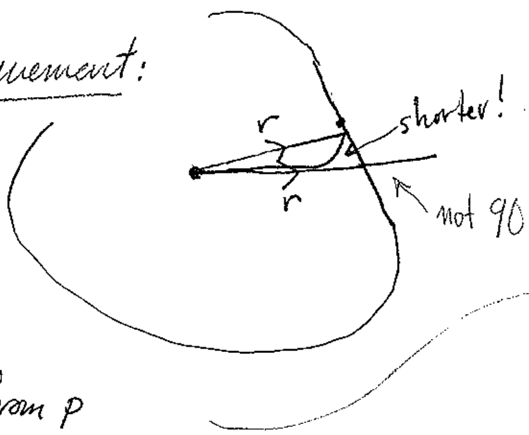
Let c be a geod through p. Then for all small r,

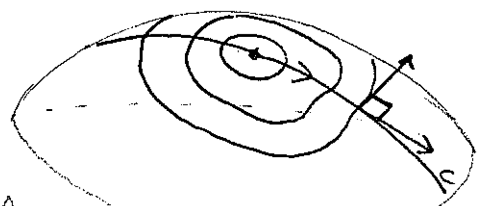
$c \perp S_r(p)$. [Tells us a lot about $D\exp_p$]



<u>Plausibility arguement:</u>







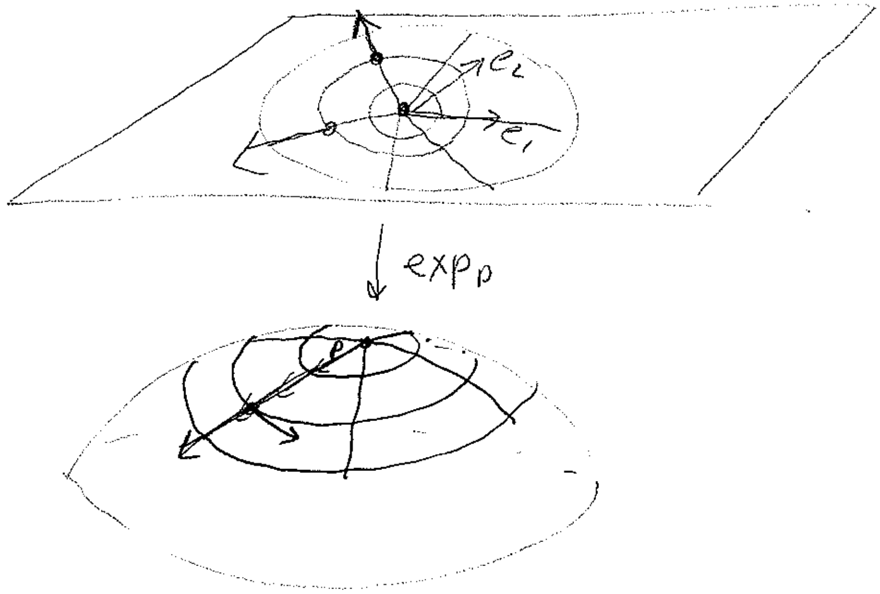
<u>Geod. Polar Coordinates</u>: around p. Choose an orthonormal basis e_1, e_2 of T_pS

$$f : (0, r_0) \times (0, 2\pi) \longrightarrow S$$

$$(r, \theta) \longmapsto \exp_p(r\sin\theta\, e_1 + r\cos\theta\, e_2)$$

For small r_0, this is a coordinate chart

[note how θ is constrained]



Key features: 28

$f((r,\theta_0)) : (0, r_0) \to S$

is a geodesic

$f((r_0,\theta)) : (0, 2\pi) \to S$

is the circle $S_r(p)$.

Metric in local coor: $g_{ij} : U \to \mathbb{R}$

$$g_{11} = 1 \text{ everywhere as } |f_r(r_0,\theta_0)|^2 = |c'_{\theta_0}(r)| = 1$$

$$\text{where } c'_{\theta_0}(r) = f((r,\theta_0)) = \exp_p\Big(r\,\underbrace{(\sin\theta_0 e_1 + \cos\theta_0 e_2)}_{\text{unit vector}}\Big)$$

$g_{12} = g_{21} = 0$ everywhere via Gauss' Lemma.

$= \langle f_r, f_\theta \rangle$

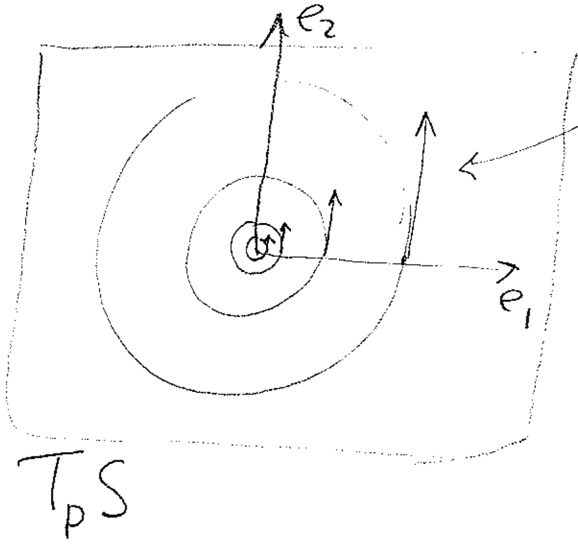
So really only one function g_{22}.

Pf of Gauss' Lemma: Will show $g_{12} = 0$.

$$\frac{\partial}{\partial r} g_{12} = \frac{\partial}{\partial r}\langle f_r, f_\theta\rangle = \underbrace{\langle f_{rr}, f_\theta\rangle}_{0 \text{ as } f_{rr} = c''_{\theta_0} \text{ is } \perp \text{ to } TS} + \langle f_r, f_{\theta r}\rangle$$

$$= \langle f_r, f_{r\theta}\rangle = \frac{1}{2}\frac{\partial}{\partial\theta}\langle f_r, f_r\rangle = 0$$

So for any fixed θ_0, have $g_{12}(r,\theta_0) = C$. Assume $\theta_0 = 0$.



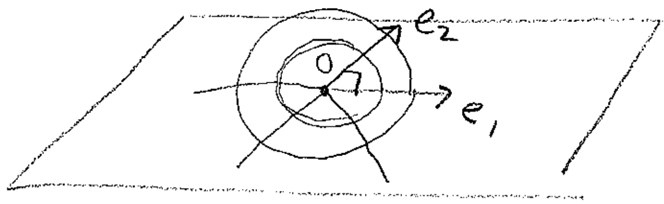
$$A(r) = \frac{g_{12}(r,\theta_0)}{2\pi r} = \left\langle f_r, \frac{f_\theta}{2\pi r} \right\rangle$$

$$= \langle D\exp_p(e_1), D\exp_p(e_2) \rangle$$

Now $A(r) = \frac{C}{2\pi r}$ and $A(0) = 0$. Thus $C = 0$ as desired. $\blacksquare$

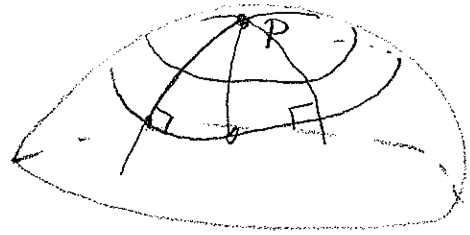
<u>Lecture 16</u>: <u>Last time</u>: geod polar coords

e_1, e_2 orthonorm. basis for $T_p S$



$f: (0, R_0) \times (-\pi, \pi)$

$$(r, \theta) \longmapsto \exp_p(r\cos\theta\, e_1 + r\sin\theta\, e_2)$$



$g_{11} = 1, \; g_{12} = g_{21} = 0$ everywhere.

<u>Intrinsic distance</u>: $d(p,q) = \inf\left\{ \text{len}(c) \;\middle|\; \begin{array}{l} c \text{ a smooth curve} \\ \text{in } S \text{ joining } p \text{ to } q \end{array} \right\}$

$S_r(p) = \exp_p\left(\begin{array}{c}\text{circle about } 0 \\ \text{of rad } r\end{array}\right) = \left[\begin{array}{l}\text{all pts joined to } p \\ \text{by geod of len } = r\end{array}\right]$

<u>Thm</u>: $p \in S \subseteq \mathbb{R}^3$. Then $\exists \varepsilon > 0$ such that

Give heuristic argument?

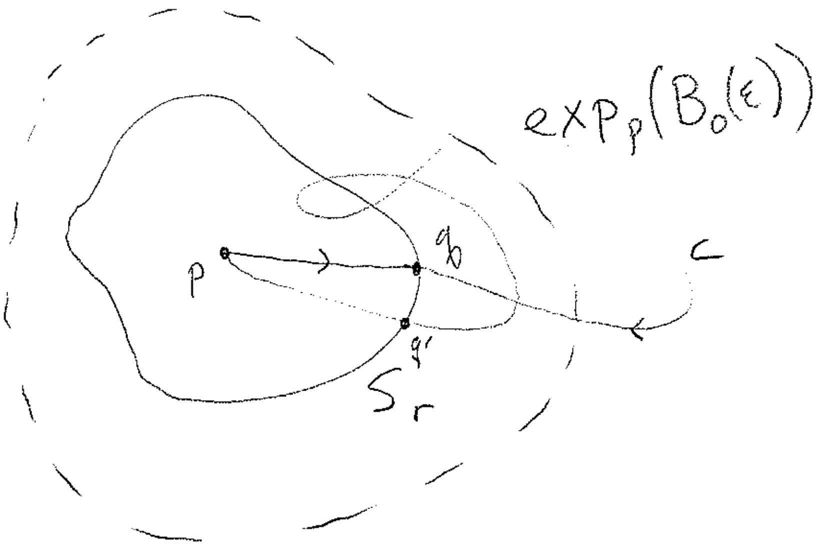
$$S_r(p) = \{q \mid d(p,q) = r\} \text{ for } r < \varepsilon.$$

and $\exists!$ geod from p to each pt of $S_r(p)$ of len r.

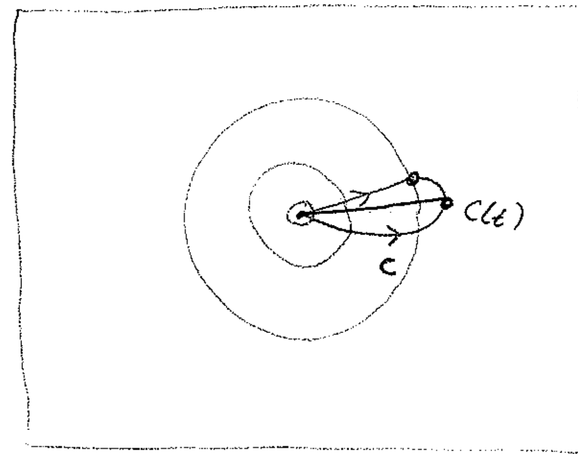
<u>Pf</u>: Choose ε s.t. $\exp_p : B_\varepsilon(0) \longrightarrow S$ is a diffeo onto its image. [Note this immediatly establishes]

Suppose $c: [0, t_0]$ be a unit speed curve joining p to $q \in S_{r_0}(p)$ where $t_0 < r_0 < \varepsilon$.

By changing q, can assume $c([0,t_0]) \subseteq \exp_p(B_\varepsilon(0))$



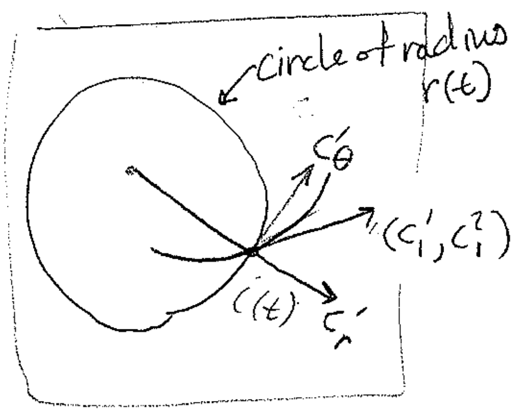
Write in local coor $(c_1(t), c_2(t))$ by exponential map



Consider $r(t) = \sqrt{c_1(t)^2 + c_2^2(t)}$

and set $(c_1', c_2') = c_r' + c_\theta'$

in polar coordinates.



Thus $|c'(t)| = |D\exp_p(c_r') + D\exp_p(c_\theta')|$

$$\geq |D\exp_p(c_r')| = |c_r'|$$

$$= |r'(t)|$$

Hence $\operatorname{len}(c) = \int_0^{t_0} |c'(t)|\, dt$

$$\geq \int_0^{t_0} |r'(t)|\, dt \geq \int_0^{t_0} r'(t)\, dt$$

$$= r(t_0) - r(0) = r_0$$

This contradicts that $\operatorname{len}(c) = t_0 < r_0$. $\blacksquare$

<u>Thm.</u> $p \in S \subseteq \mathbb{R}^3$. Then $\exists\, \varepsilon > 0$ and U open nbhd of p s.t. $\forall\, q_1 \in U$, $\exp_{q_1}|_{B_\varepsilon(0)}$ is a diffeo onto its image. If $q_2 \in S$ and $d(q_1, q_2) < \varepsilon$ then $\exists$ a unique geod from q_1 to q_2 whose length is $d(q_1, q_2)$.

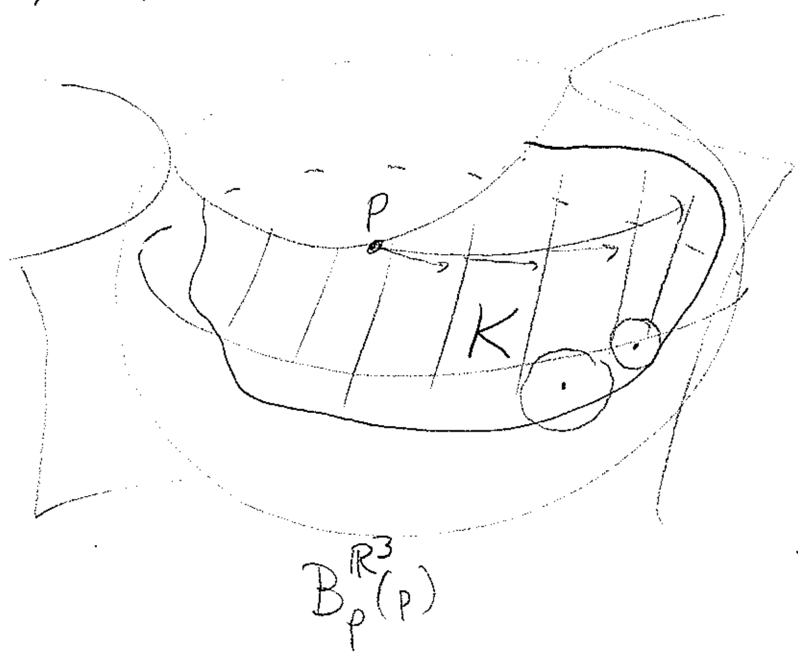
Pf: First sentence follows from O.D.E. theory see Prop 8.2.3. Rest follows from preceding. ∎

<u>Thm</u>: Suppose $S \subseteq \mathbb{R}^3$ is a smooth surface which is <u>closed</u> in $\mathbb{R}^3$. Then $\forall p$, $\exp_p$ is defined on all of T_pS.

Pf: Suppose not, and there is p and a unit vector $v \in T_pS$.

$$\rho = \sup \{ t_0 \mid \exists \text{ a geod } c : (\delta, t_0) \to S \text{ with } c(0) = p, c'(0) = v \}$$

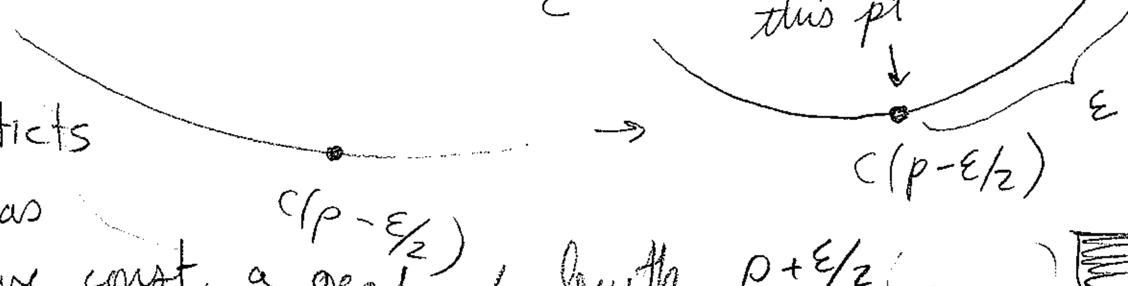
is $< \infty$. Let $c : (\delta, \rho) \to S$ be the maximal geod w/ $c'(0) = v$, $c(0) = p$. Note that $\text{image}(c) \subseteq B_\rho^{\mathbb{R}^3}(p)$.



Let $K = S \cap \overline{B_\rho(p)}$, with is compact.

By Thm, $\exists\, \varepsilon$ s.t. $\forall q_1 \in K$, $\exp_{q_1}|_{B_\varepsilon(0)}$ is a diffeo.

Look at c at time $\rho - \varepsilon/2$. By ↑ we can extend c at least ε beyond this pt



This contractices def of ρ, as have const. a geod of length $\rho + \varepsilon/2$. ∎

Lecture 17: Last time: short geodesics minimize length

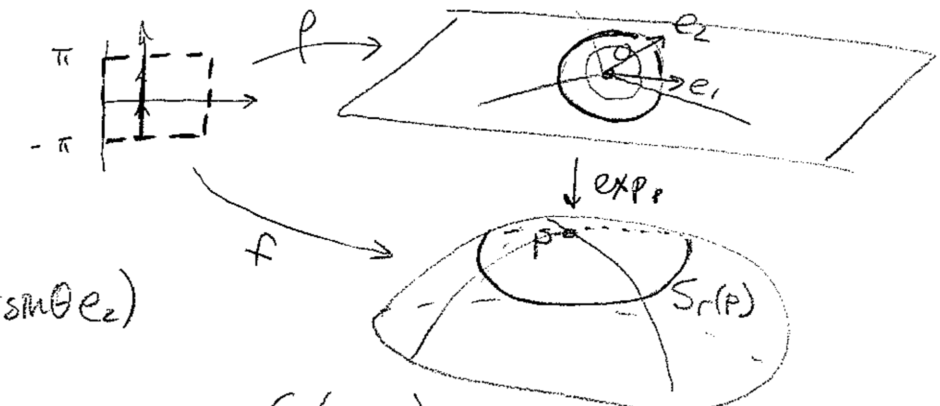
Today: $\text{length}(S_r(p)) = 2\pi r\left(1 - \frac{K(p)}{6} r^2 + O(r^3)\right)$

Geodesic polar coords: Fix orthonormal basis e_1, e_2 for T_pS.

$f: (0, R_0) \times (0, 2\pi)$

$\downarrow$

$\exp_p(r\cos\theta\, e_1 + r\sin\theta\, e_2)$



$\underbrace{g_{11} = 1,\ g_{12} = g_{21} = 0}_{[\text{constant}]} \quad g_{22} = G(r,\theta).$

$\text{length}(S_r(p)) = \int_0^{2\pi} \sqrt{G(r,\theta)}\, d\theta$

Calculating $K(r,\theta)$

$\{f_r, f_\theta, n\}$

HW

$f_{rr} = \quad - L_{11} n$

$\frac{\partial G}{\partial r}$

$f_{\theta r} = \frac{1}{2}\frac{G_r}{G} f_\theta - L_{21} n$

where (L_{ij}) is the Gauss map written w.r.t. f_r, f_θ

$n_r = L_{11} f_r + L_{12} f_\theta$

$n_\theta = L_{21} f_r + L_{22} f_\theta$

Look at $f_{rr\theta} = f_{\theta rr}$ and equate f_θ components.

$-L_{11}L_{22} = \underbrace{\frac{1}{2}\left(\frac{G_r}{G}\right)_r + \frac{1}{4}\left(\frac{G_r}{G}\right)^2}_{} - L_{21}L_{12}$

$\Rightarrow -K = -\det(L_{ij}) = \frac{(\sqrt{G})_{rr}}{\sqrt{G}}$

should be L_{12}

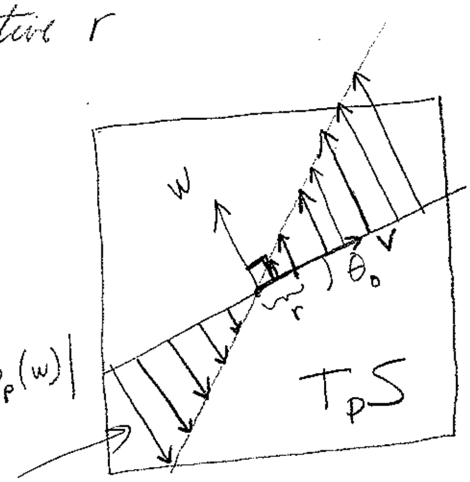
Fix θ_0, consider $\alpha_{\theta_0}(r) : (-R_0, R_0) \longrightarrow \mathbb{R}$

given by $\alpha_{\theta_0}(r) = \sqrt{G(r,\theta_0)}$ for positive r

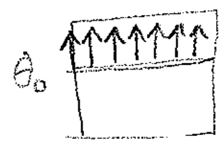
$V = \cos\theta_0 e_1 + \sin\theta_0 e_2$

$W = -\sin\theta_0 e_1 + \cos\theta_0 e_2$

$$\alpha_{\theta_0}(r) = \left| D_{rv} \exp_p (rw) \right| = r \left| D_{rv} \exp_p(w) \right|$$



Note: These two defs agree,



and $f = \exp \circ \rho$

$G(r,\theta_0) = D_{(r,\theta_0)} f(0,1)$

$\rho_\theta = D_{(r,\theta_0)} \rho(0,1) = rw$

However, $\alpha_{\theta_0}(r)$ is smooth as $= r \underbrace{\left| D_{rv} \exp_p(w) \right|}_{\text{involves } \sqrt{\ } \text{ but only near } 1.}$

$$\alpha_{\theta_0}(0) = 0$$

$$\alpha'_{\theta_0}(0) = \Big(|D \exp_p(w)| + r \, |D \exp|' \Big)\Big|_{r=0} = 1$$

$$\alpha''_{\theta_0}(0) = \lim_{r \searrow 0} \left(\alpha''_{\theta_0}(r) = \sqrt{G(r,\theta_0)}_{rr} = -K(r,\theta_0)\,\alpha(r) \right)$$

$$= 0$$

$$\alpha'''_{\theta_0}(0) = \lim_{r \searrow 0} \left(\alpha'''_{\theta_0}(r) = -K(r,\theta_0)' \cdot \alpha(r) - K(r,\theta_0)\,\alpha'(r) \right)$$

$$= -K(p)$$

So: $\alpha_{\theta_0}(r) = r - \frac{K(p)}{6} r^3 + O(r^4)$

Recall $f(r)$ is $O(r^4)$ if $\exists C$ s.t. $|f(r)| \le C r^4$

$$\text{Length}(S_r(p)) = \int_0^{2\pi} \alpha_\theta(r)\, d\theta = 2\pi r\left(1 - \frac{K(p)}{6} r^2 + O(r^3)\right)$$

[Query: how did I cheat?] $\alpha_\theta(r) = r - \frac{K(p)}{6} r^3 + \underbrace{E_\theta(r)}_{\text{depends on } \theta}$

Aside: $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & x \neq 0 \\ 0 & x = 0 \end{cases}$

along every ray $f(r,\theta_0) = O(r)$

$f(y^2, y) = \frac{1}{2}$ for all y

Where does $E_\theta(r)$ come from?

$$E_\theta(r) = \frac{1}{4!}\alpha_\theta''''(r_0)$$

$r_0 \in (0, r)$

$$\alpha_\theta''''(r_0) = -2K(r,\theta)_r$$

Think of $K(\exp_p)$ as a fn of a pt in T_pS

On a cpt set, any directional derivative in a unit direction is bounded

$$K(r,\theta)_r = aK(r,\theta)_{e_1} + bK(r,\theta)_{e_2} \quad v = ae_1 + be_2$$

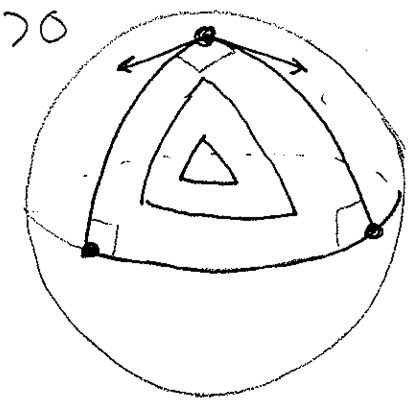
Thus error term is uniformly under control so formula for len. holds. So Gauss curvature is intrinsic. Q.E.D.

Lecture 18: Last time: curvature is intrinsic

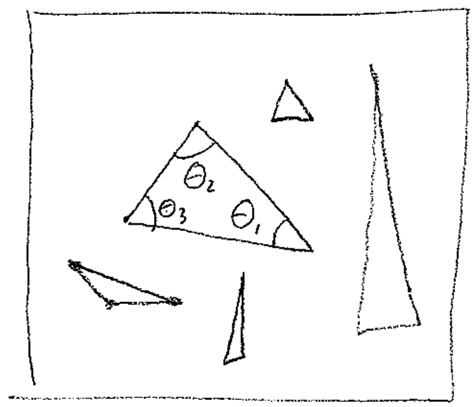
Today: Gauss-Bonnet: [local and global.]

local version: geodesic triangles.

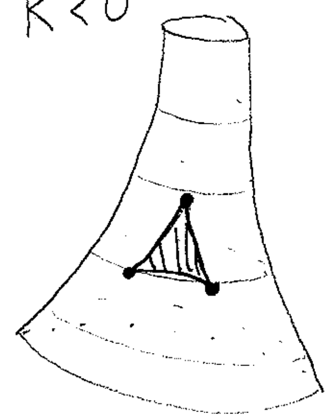
$K > 0$



$K = 0$

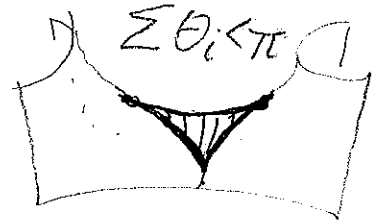


$K < 0$



$\sum \theta_i = \frac{3}{2}\pi$ for unit Δ, $\quad \sum \theta_i = \pi$

$\sum \theta_i > \pi$



Thm: $S \subseteq \mathbb{R}^3$ smooth surface. T in S a geod. Δ. with interior angles $\theta_1, \theta_2, \theta_3$. Then

$$\sum \theta_i - \pi = \int_T K \underbrace{dA}_{\text{area}}$$

[Point out reasonableness w.r.t. scaling.]

Where $\int$ means: Suppose $f: U \xrightarrow{\text{chart}} S$ w/ $f(U) \supseteq T$

$$\text{Area}(T) = \iint_{f^{-1}(T)} \sqrt{g_{ij}}\, dx\, dy \qquad [g_{ij} \text{ metric coeffs}]$$

Suppose $\rho: S \to \mathbb{R}$ some cont fn

$$\int_T \rho \, dA = \iint_{f^{-1}(T)} \rho \circ f \sqrt{g_{ij}}\, dx\, dy$$

For regions not contained in a chart, break into pieces integrate over each piece, add the result. [as w/ area, doesn't matter how we subdivide.]

<u>Gauss-Bonnet</u>: Suppose $S \subseteq \mathbb{R}^3$ is a compact smooth surface. Then $\int_S K \, dA = 2\pi \chi(S)$

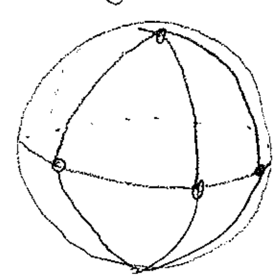
[Discuss why this is surprising.]

<u>Cor</u>: Suppose S is a cpt surface in $\mathbb{R}^3$ w/ $K > 0$ everywhere. Then $S \cong S^2$

<u>Pf</u>: Gauss Bonnet + P doesn't embed in $\mathbb{R}^3$.

<u>Pf that local $\Rightarrow$ global.</u>

A triangulation of S is <u>geodesic</u> if every edge is a geodesic segment. <u>Ex</u>.



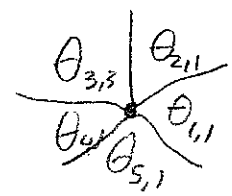
<u>Thm 8.4.1</u>: Any cpt surface S in $\mathbb{R}^3$ has a geodesic triangulation.

[constructed locally at small scales]

Let $T_1, \dots, T_n$ be the triangles of a geod tri of S

$$\int_S K\,dA = \sum_{i=1}^{n} \int_{T_i} K\,dA \overset{\text{local}}{=} \sum_{i=1}^{n} (\theta_{i,1} + \theta_{i,2} + \theta_{i,3} - \pi)$$

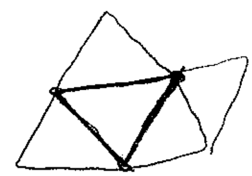
$$= \sum \begin{pmatrix}\text{interior angles} \\ \text{of } T_i\end{pmatrix} - \pi n$$



$$= 2\pi(\#\text{ of verts}) - \pi(\#\text{ triangles})$$

$$= 2\pi\left(v - \tfrac{1}{2}f\right) = 2\pi\chi(S)$$

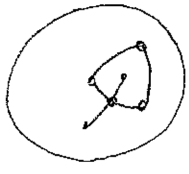
[Query] Each triangle contributes 3 edges, double counting, so $e = \frac{3}{2}f$



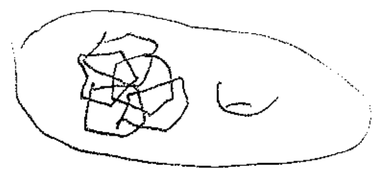
$v - e + f = v - \frac{1}{2}f$ ✓

<u>Proof of 8.4.1:</u> <u>Key idea:</u> $B = \exp_p(B_\varepsilon(0))$ is geodesically convex for small ε, i.e.

any two pts in B are joined by a unique minimal geod arc which <u>lies in B</u>. Any two such geod intersect in at most one pt.



<u>Then</u> cover S w/ geod polygons.

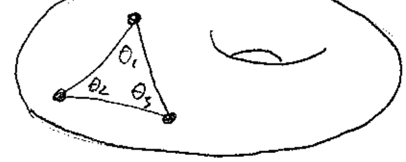


Then triangulate the comp regions, as in Euclidean space.

Lecture 19: Last time: Gauss-Bonnet: S cpt $\Rightarrow \int_S K\,dA = 2\pi\chi(S)$

Follows from: Thm: $T \subseteq S$ a geod triangle.

Then $\int_T K\,dA = \sum \theta_i - \pi$



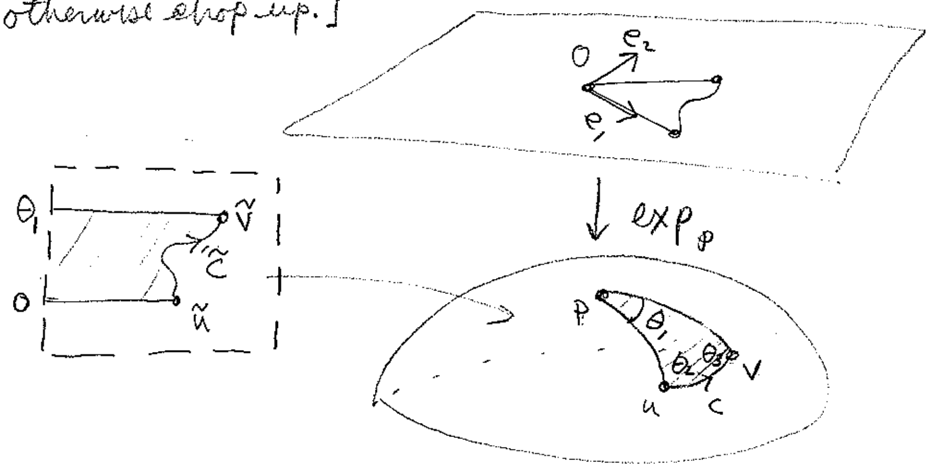
Today: Proof of Thm.

Assume that T is small enough to be contained in geod. polar coor patch. [otherwise chop up.]

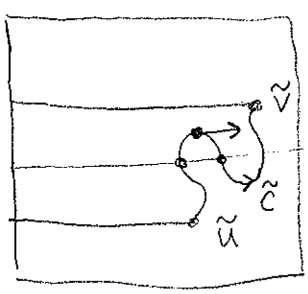
$f: (0,R_0)\times(-\delta, 2\pi-\delta) \to S$

$f(r,\theta) = \exp_p(r(\cos\theta e_1 + \sin\theta e_2))$

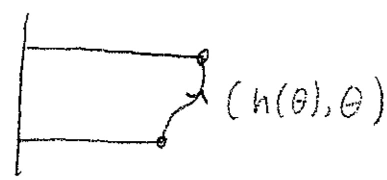
$\tilde{u} = f^{-1}(u)$ $\tilde{v} = f^{-1}(v)$



Claim 1: $\tilde{c}$ intersects each line $(*, \theta_0)$ in at most one pt.



Suppose not. By IVT, $\exists$ a point where $\tilde{c}'$ is horizontal. [Query] contradicts uniqueness of geodesics



Thus, $\tilde{c}$ can be param by $(h(\theta),\theta)$ for $\theta \in (0,\theta_0)$.

$$\int_T K\,dA = \int_0^{\theta_1}\int_0^{h(\theta)} K(f(r,\theta))\sqrt{\det(g_{ij})}\,dr\,d\theta$$

$g_{11} = 1$

$g_{12} = g_{21} = 0$

$g_{22} = G$

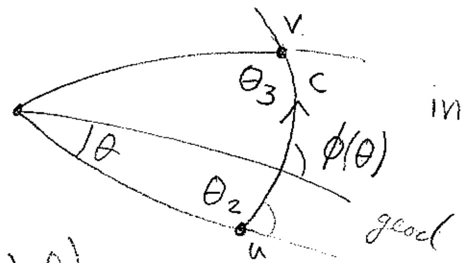
$$-K = \frac{(\sqrt{G})_{rr}}{\sqrt{G}}$$

$$= \int_0^{\theta_1}\int_0^{h(\theta)} -\sqrt{G}_{rr}\; dr\, d\theta = -\int_0^{\theta_1} \sqrt{G}_r \Big|_{r=0}^{h(\theta)} d\theta$$

$$= \int_0^{\theta_1} 1 - (\sqrt{G})_r(h(\theta),\theta)\, d\theta = \theta_1 + \int_0^{\theta_2} -(\sqrt{G})_r(h(\theta),\theta)\, d\theta$$

— at least, this looks good!

Define $\phi(\theta)$ to be



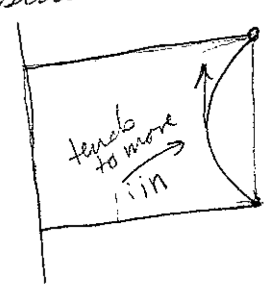
in S

<u>Claim:</u> $\phi'(\theta) = -(\sqrt{G})_r(h(\theta),\theta)$

Assuming this get

$$\int_T K\, dA = \theta_1 + \phi\Big|_0^{\theta_1} = \theta_1 + \overbrace{\phi(\theta_1)}^{\theta_3} - \overbrace{\phi(0)}^{\pi-\theta_2} = \Sigma\theta_i - \pi$$

Consider



$\tilde{c}(\theta) = (h(\theta),\theta)$
$c = f\circ\tilde{c}$

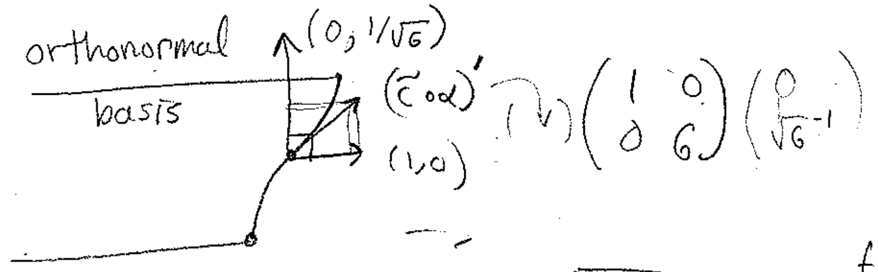
Choose $\alpha : \underset{s}{[0,\varepsilon]} \to [0,\theta_0]$ so that $c\circ\alpha$ is unit speed ($\Rightarrow$ geodesic)

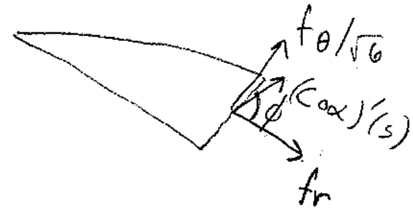
$\tilde{c}\circ\alpha(s) = (h\circ\alpha(s), \alpha(s))$

Geodesic equation (using Γ^i_{jk} from HW) for 1st coor give:

$$(h\circ\alpha)'' = \underbrace{\frac{1}{2} G_r(\tilde{c}\circ\alpha(s))}_{=\,\Gamma^1_{22}}(\alpha'(s))^2$$

Let's compute ϕ:





$$\boxed{\cos(\phi\circ\alpha(s)) = f_r \cdot (c\circ\alpha)' = (h\circ\alpha)'(s)}$$

$$\boxed{\sin(\phi\circ\alpha(s)) = \cos(\pi/2 - \phi\circ\alpha(s)) = \frac{f_\theta}{\sqrt{G}}\cdot(c\circ\alpha)' = \sqrt{G}\,\alpha'(s)}$$

Thus $\frac{1}{2} G_r(\tilde{c}\circ\alpha(s))\,\underline{(\alpha'(s))^2} = (h\circ\alpha'')(s)$

$$= (\cos(\phi\circ\alpha(s)))'(s) = -\sin(\phi\circ\alpha(s))\,\phi'(\alpha(s))\cdot\alpha'(s)$$

$$= -\sqrt{G(\tilde{c}\circ\alpha(s))}\cdot\phi'(\alpha(s))\,\underbrace{(\alpha'(s))^2}_{\text{cancel}}$$

$$\phi'(\alpha(s)) = -\frac{1}{2}\,\frac{G_r(h\circ\alpha(s),\alpha(s))}{\sqrt{G(h\circ\alpha(s),\alpha(s))}} = -\sqrt{G}_r(h\circ\alpha(s),\alpha(s))$$

as desired. Q.E.D.

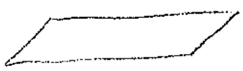
Lecture 20: Geometry of abstract surfaces.

[Problematic things:] Nice: $\int_{S^2} K\, dA = 4\pi$ also $\exists$ a sphere w/ $K = 1$ everywhere

Bad: $\int_{T^2} K\, dA = 0$ but D.N.E. a T^2 in $\mathbb{R}^3$ w/ $K = 0$ everywhere.

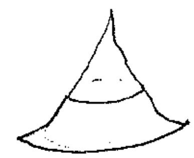
Also

$K = 0$



$K = 1$

but no closed surface of curv $K = -1$ [Hilbert]



Def. Let $S \subseteq \mathbb{R}^n$ be a smooth surface. A Riemannian metric I on S is a family of pos def sym bilinear forms

$I_p : T_pS \times T_pS \longrightarrow \mathbb{R}$ which is smooth. $\quad I_p(v,v) > 0 \quad v \neq 0$

$I_p(v,w) = I_p(w,v)$

smooth means: $\forall$ coor chart $f: U \to S$ let

$g_{ij}(p) = I_p(f_i(p), f_j(p))$. Then g_{ij} is a smooth fn.

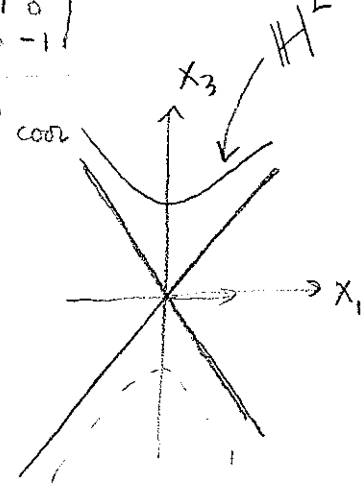
Ex: Hyperboloid model for the hyperbolic plane.

$x, y \in \mathbb{R}^3$, $\langle x, y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

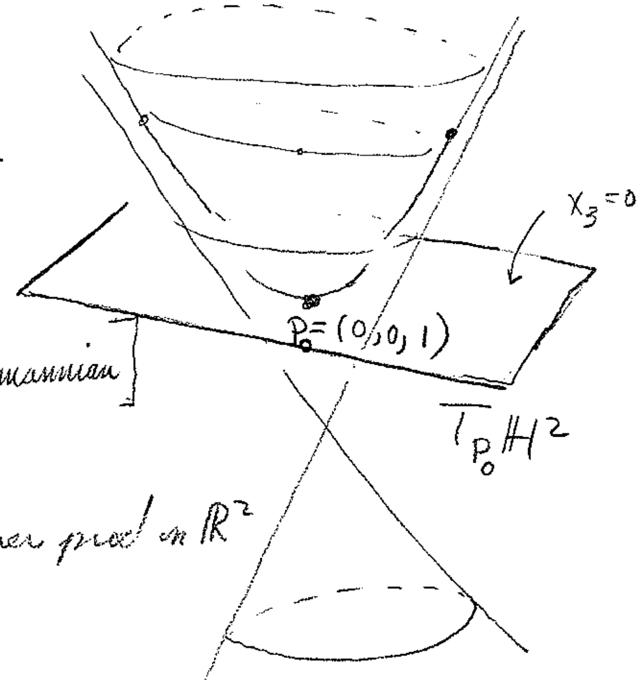
$\mathbb{H}^2 = \{ x \in \mathbb{R}^3 \mid \langle x, x \rangle = -1,\ x_3 > 0 \}$ $\quad$ Ignore 2^{nd} coor

$\langle x, x \rangle = 0 \qquad x_1^2 = x_3^2 \Rightarrow x_1 = \pm x_3$

$\langle x, x \rangle = -1 \qquad x_1^2 - x_3^2 = -1$



Topologically this is just $\mathbb{R}^2$.



For $p \in \mathbb{H}^2$, define $I_p : T_p\mathbb{H}^2 \times T_p\mathbb{H}^2 \to \mathbb{R}$

by $I_p(v,w) = \langle v, w \rangle$ ← Lorentzian

[Query: What do we need to check to see this is Riemannian]

Consider $P_0 = (1,0,0)$. There $I_{P_0} =$ usual Euclid inner prod on $\mathbb{R}^2$

So are ok here.

Set $O(2,1) = \{ A \in GL_3\mathbb{R} \mid \langle Ax, Ay \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{R}^3 \}$

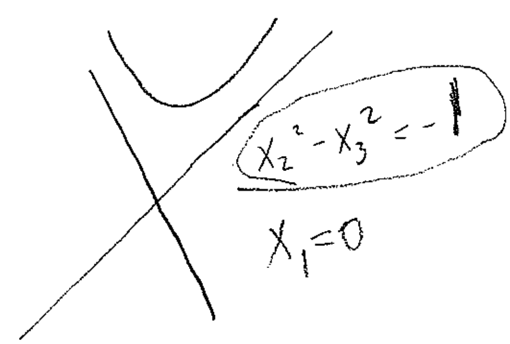
["isometries of $\langle , \rangle$"] $\quad O_0(2,1)$ those which pres $\mathbb{H}^2$

$\boxed{SO_0(2,1) \text{ those w/ det } 1.}$ ← don't really need to mention.

<u>Claim</u>: $SO_0(2,1)$ acts transitively on $\mathbb{H}^2$, i.e. given $x, y \in \mathbb{H}^2$, $\exists A \in SO_0(2,1)$ such that $Ax = y$.

<u>Cor</u>: I_p is always pos def $\Rightarrow (\mathbb{H}^2, I_p)$ is a Riemannian surface.

<u>Pf</u>: • Any $M \in O(2)$ gives an element via $\begin{pmatrix} M & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}$



• $\begin{pmatrix} 1 & 0 & 0 \\ 0 & x_3 & x_2 \\ 0 & x_2 & x_3 \end{pmatrix}$ takes $(0,0,1)$ to $(0, x_2, x_3)$

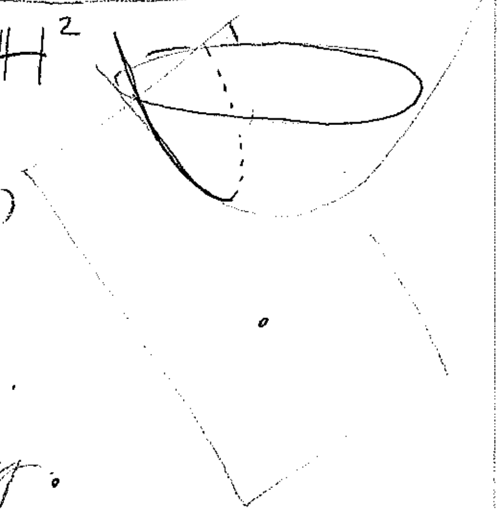
$x_3x_2 - x_2x_3 = 0$

$1, \quad x_3^2 - x_2^2 = 1,$

Note: $A \in O_0(2,1)$ gives an isometry of $(\mathbb{H}^2, I_p)$.

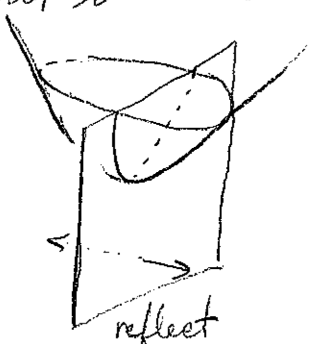
Meta-Def: Any intrinsic notion encountered earlier is defined for a Riemannian surface in the same way (usually via local coordinates).

Geodesics in $\mathbb{H}^2$: $\gamma = (\text{Plane through } 0) \cap \mathbb{H}^2$



Pf: Since elts $A \in O_0(2,1)$ take planes (through 0) to planes (through 0) and it acts transitively it is enought to check this for planes through p_0.

But this is clear using the reflection symmetry.



$$\begin{pmatrix} R & 0 \\ & 0 \\ 0\ 0 & 1 \end{pmatrix}$$

Curvature: As can move any pt to any other via an isometry, $K =$ constant

[$K > 0$ implausible; $K = 0$ should imply that $\mathbb{H}^2$ is isometric to $\mathbb{R}^2$, but it violates the parallel postulate.]

HW: In fact $K = -1$.

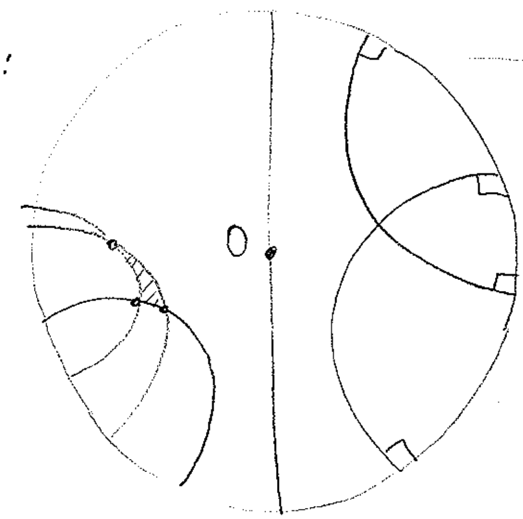
Annother point of view: <u>Poincare Disc Model</u>

$$D = \{ z \in \mathbb{C} \mid |z| < 1 \} \qquad I_p(v,w) = \frac{4}{(1-r^2)^2} \underbrace{V \cdot W}_{\text{usual Euclidean inner product}}$$

$$r = \sqrt{x^2+y^2}$$

Angles same but dist distorted

<u>Geodesics:</u>



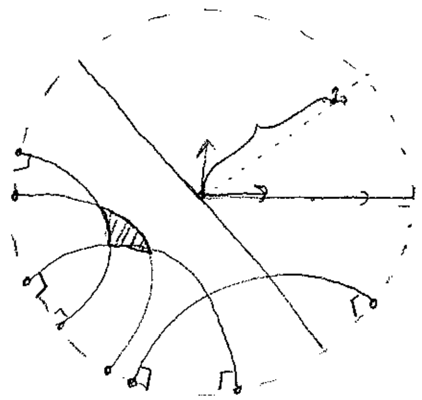
Geodesics are circles meeting ∂D in right angles (+ straight lines through 0 as a special case.

<u>Note:</u> Boundary is infinitely far away

<u>Note:</u> Is isometric to earlier example.

Lecture 21: Hyperbolic plane $\begin{cases} \text{hyperboloid model (Last time)} \\ \text{Poincaré disc model (today)} \end{cases}$

Poincaré Model: $D = \{|z| < 1 \mid |z| < 1\}$



$$I_p(v,w) = \frac{4}{(1-r^2)^2} \overbrace{V \cdot W}^{\text{Usual Euclidean dot prod}}$$

$$r = \sqrt{x^2 + y^2}$$

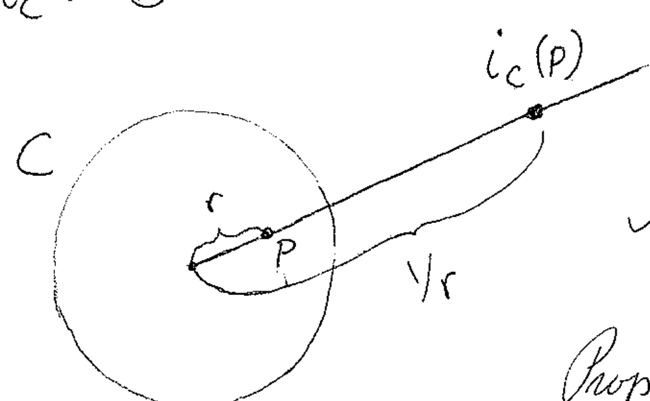
Distances are distorted but not angles. Boundary is infinitely far away.

Geodesics: circles meeting boundary in right angles (w/ straight lines through 0 as a special case.)

[Can prove these are geodesics through symmetry, but understanding the others will take some work.]

Inversions: C a circle in $\mathbb{C}$.

"Inversion in C": $i_C : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ $\qquad \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1 = S^2$



Also (center $\xleftrightarrow{i_C} \infty$)

Prop: $i_C \circ i_C = id$

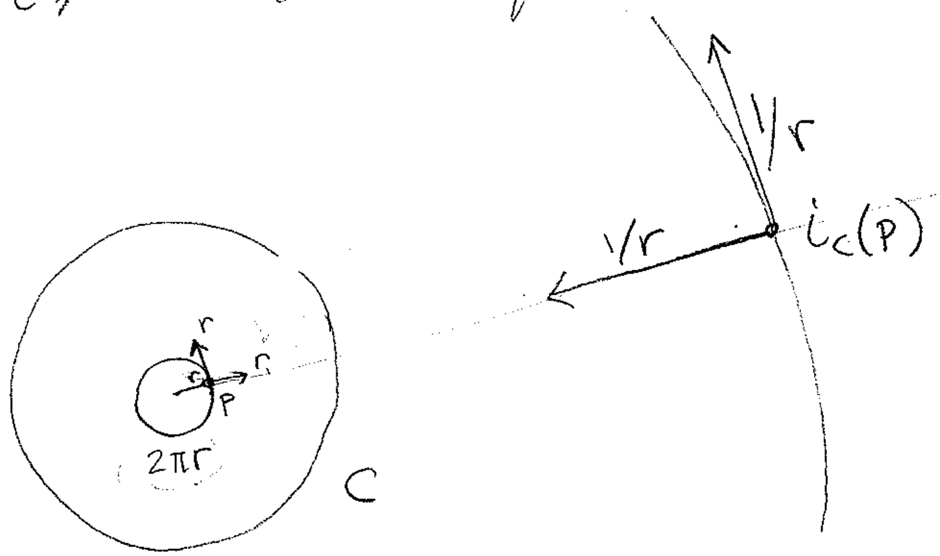
Lemma: Under i_C: • circles not containing the center of C go to circles not containing the center

• circles through the center go to lines • lines go to lines or circles.

Pf: Check alg, starting w/ unit circle about O, noting $i_C = (z \mapsto 1/\bar{z})$ and using complex notation, e.g. $C = \{ z \mid |z - c_0|^2 = r_0^2 \}$.

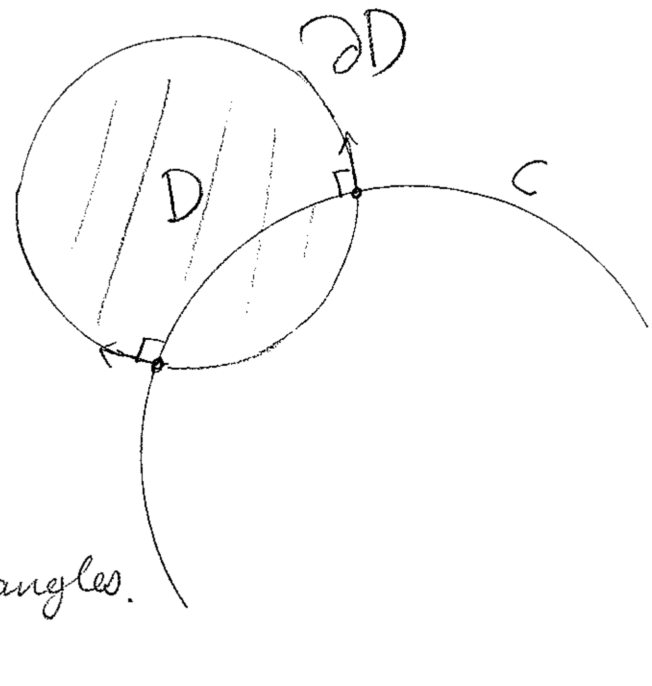
Lemma: i_C preserves angles (is conformal)

Pf:



Back to Poincaré model:

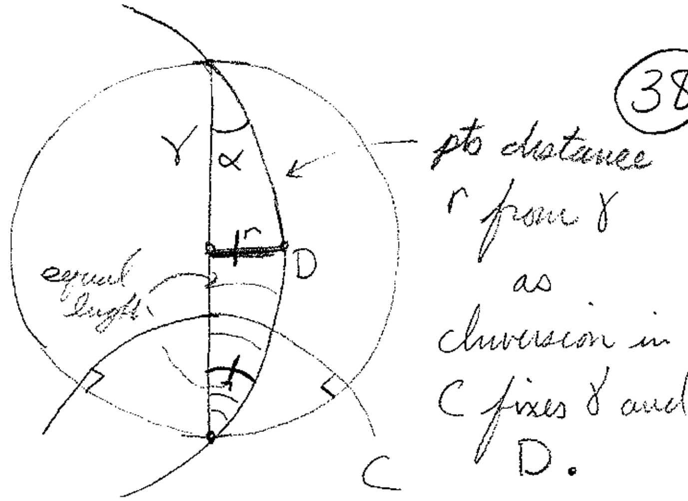
If $C \perp$ to ∂D, then i_C preserves D.



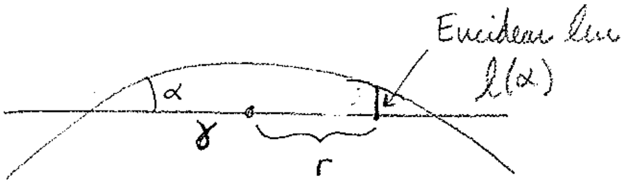
Claim: i_C is an isom of (D, I_P) ($\Longrightarrow$ geodesics are as claimed)

Good sign: i_C preserves angles.

Pf. of Claim: Calculation.



Plausibility Arguement:



$$\sqrt{I_p(\text{Euc}_{\text{unit}})} = \left(\frac{dl}{d\alpha}\right)^{-1} = \frac{2}{1-r^2}$$

$\boxed{\text{Probably skip}}$

Facts:

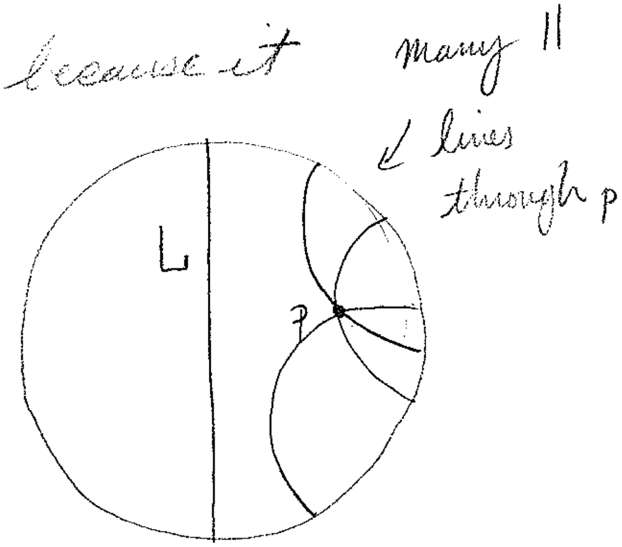
$\text{Isom}((D, I_p))$ is generated by inversions.

$\text{Isom}^+(D, I_p)$ = biholomorphic maps from D to itself

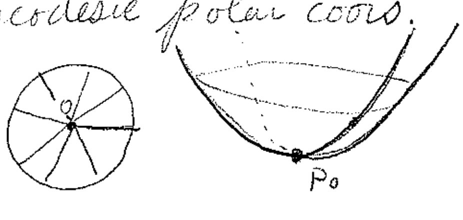
orientation preserving

$$= \left\{ z \mapsto e^{i\theta}\left(\frac{z-\alpha}{-\bar{\alpha}z+1}\right) \;\middle|\; \begin{matrix}\theta \in \mathbb{R} \\ \alpha \in D\end{matrix} \right\}$$

This is really the same as what we looked at last time — $\text{Isom}((D, I_p))$ is transitive and it can't be the Euclidean plane because it violates the parallel postulate:

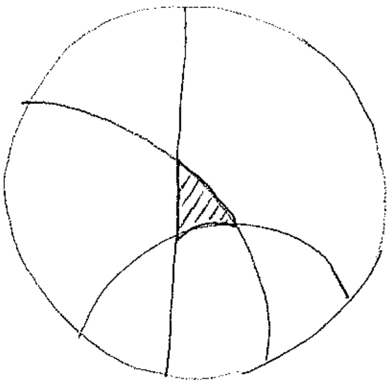


Explicitly: Construct a map using geodesic polar coors.

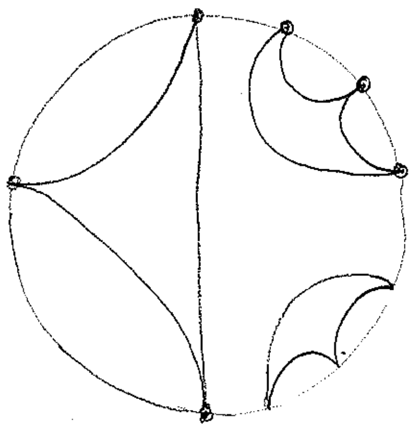


Triangles: $\underbrace{\int_T K \, dA}_{= -\text{Area}} = \theta_1 + \theta_2 + \theta_3 - \pi$ $\qquad K = -1$

$\Rightarrow \text{Area} = \pi - \theta_1 - \theta_2 - \theta_3 \Longrightarrow$ Every triangle has area less than π!

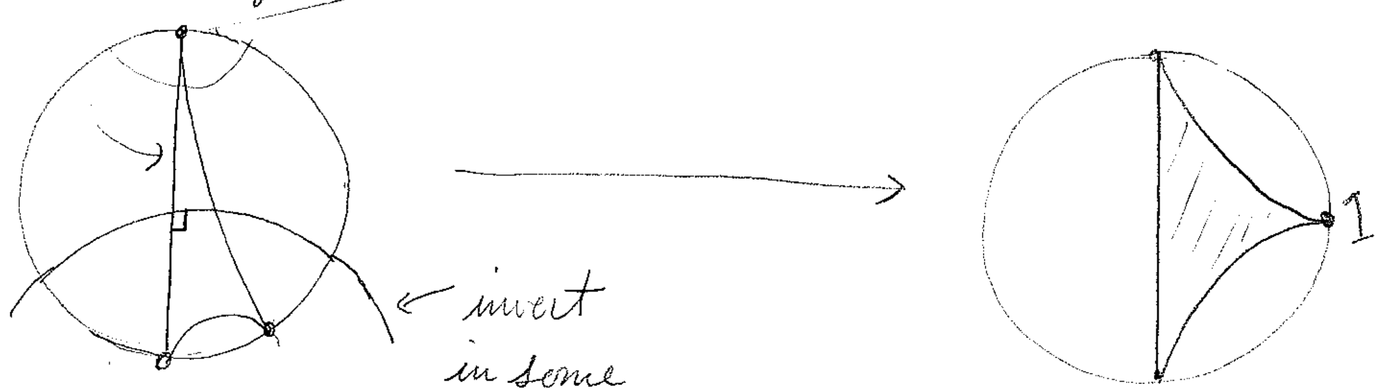


Ideal "triangles": all "vertices" at ∞



These are all isometric for the following reason.

Take one edge to



$\leftarrow$ invert in some circle like this

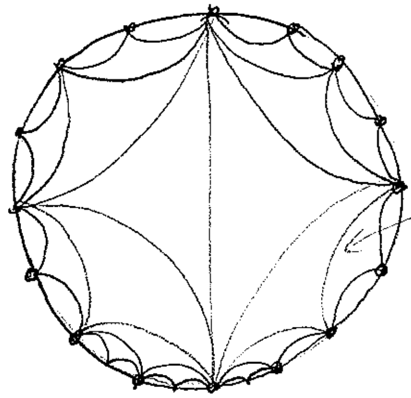
These all have area π (HW).

Lecture 22: Last time: Poincaré model.

Today: Tilings of hyperbolic plane, hyp metrics on cpt surfaces.

$$I_p(v,w) = \frac{4}{(1-r^2)^2}\, v \cdot w$$

Ideal triangles:



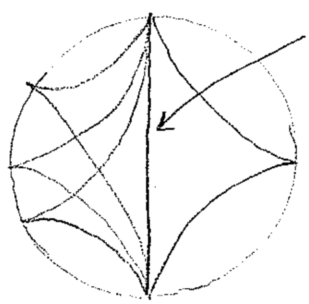
Can do this symmetrically, i.e.

Given two triangles 

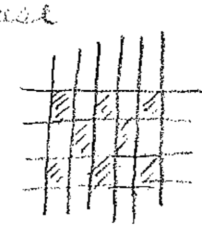
$\exists$ an isom $g \in \mathrm{Isom}(\mathbb{H}^2)$ taking T_1 to T_2 preserving the whole tiling.

[The one I drew is not symmetric, hand out one that is.
Query to nature of the difference.]

Issue:



Like in Euclidean case



vs. Penrose tiling

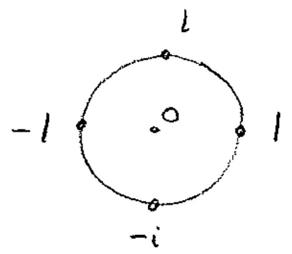
Upper Halfspace Model. $H = \{ z \in \mathbb{C} \mid \mathrm{Re}\, z > 0 \}$

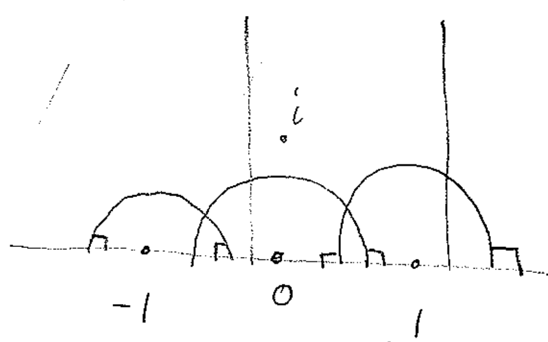
$$I_p(v,w) = \frac{1}{y^2}\, v \cdot w \qquad p = x+iy$$

An Isometry:

$D \longrightarrow H$

$$z \longmapsto \frac{z+i}{iz+1}$$





geodesics are still circles meeting ∂ in $\perp$.

$$\mathrm{Isom}^+(H) = \left\{ z \longmapsto \frac{az+b}{cz+d} \;\middle|\; \underbrace{a,b,c,d \in \mathbb{R}}, \; ad \neq bc \right\}$$

can rescale all simult. without changing the result Möbius transformation.

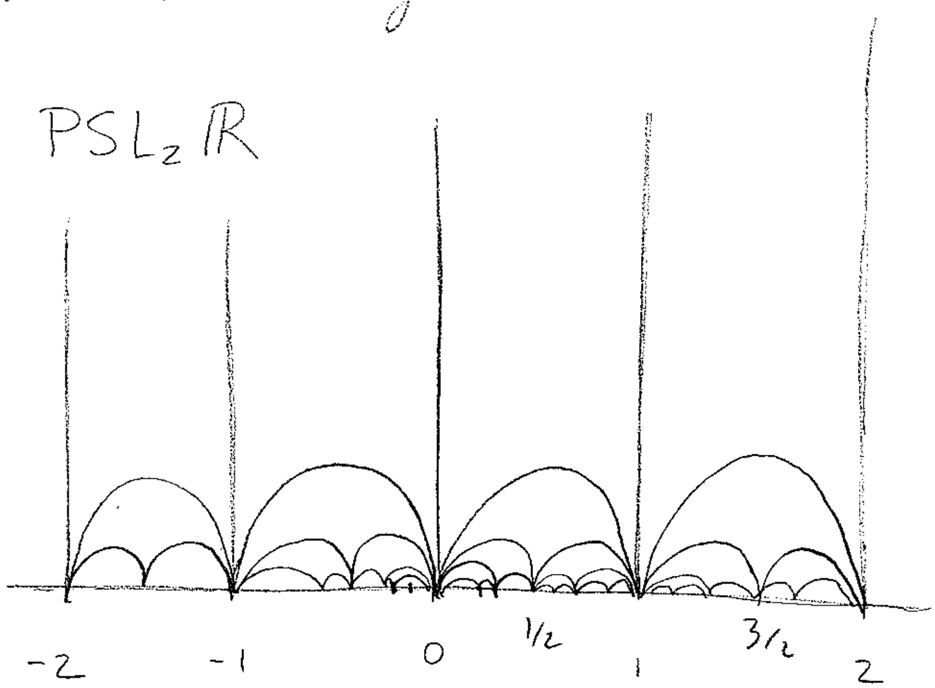
$$PSL_2\mathbb{R} = SL_2\mathbb{R} \big/ \{\pm I\} \xrightarrow{\cong} \mathrm{Isom}^+(H)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(z \to \frac{az+b}{cz+d} \right)$$

What does this have to do with tilings?

$$\Gamma = \underset{\text{interger entries}}{\underline{PSL_2\mathbb{Z}}} \leq PSL_2\mathbb{R}$$

Preserves this tiling:



$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$$

Apply to $p/q \in \mathbb{Q}$

$$p/q \longmapsto \frac{a\,p/q + b}{c\,p/q + d} = \frac{ap+bq}{cp+dq} = r/s \text{ where } \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

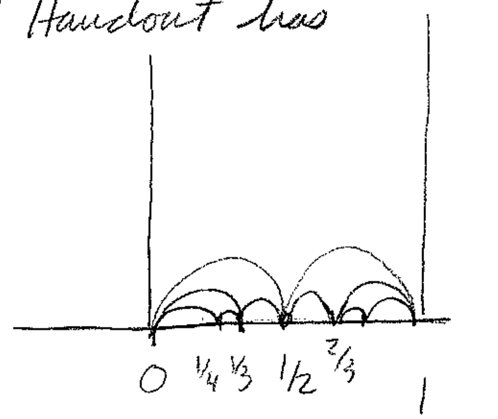
$$p/q \longleftrightarrow \begin{pmatrix} p \\ q \end{pmatrix}$$

in lowest terms

Now draw a geod from p_1/q_1 to p_2/q_2 if $|p_1 q_2 - p_2 q_1| = 1$.

This is preserved by A as $|p_1 q_2 - p_2 q_1|$ is $\det\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}$ and taking det of $\begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix} = A\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}$ gives the desired result.

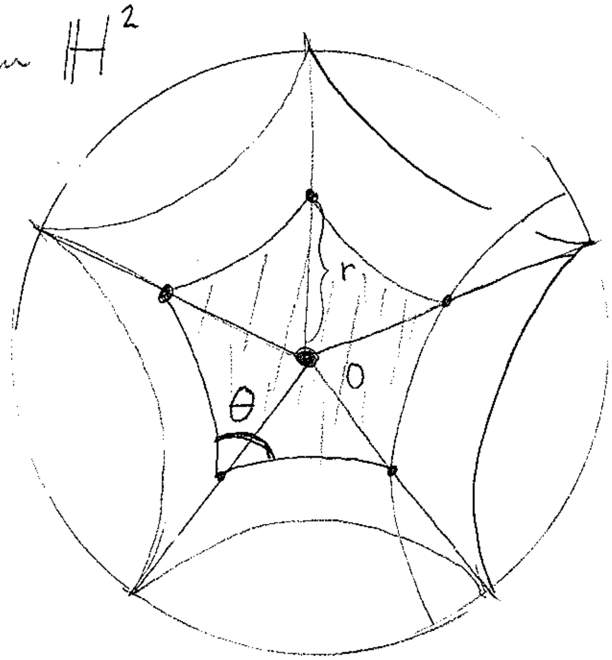
This is the triangulation shown above (Handout has the same in the disc model.)



Mention connection to modular forms, # theory etc.

What about bounded tiles:

<u>Lemma:</u> $\exists$ a right angle pentagon in $\mathbb{H}^2$



Pf: Use disc model.

Consider the pentagon $P(r)$ shown, and consider $\theta(r)$

For small r, $P(r)$ is nearly Euclidean, hence $\theta(r) = 2\pi/5$

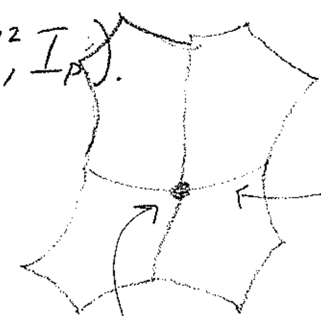
Also $\theta(1) = 0$. By continuity, $\exists\, r_0$ w/ $\theta(r) = \pi/2$

Can tile $\mathbb{H}^2$ with such pentagons, in a necess. sym. fashion. See Handout. Two ways to prove:

1) Write down group explicitly.

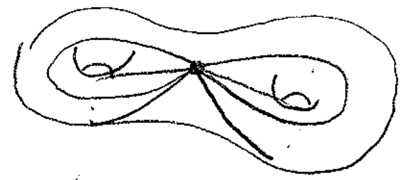
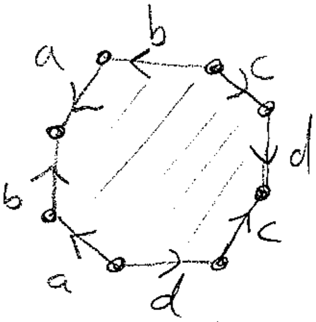
2) Use:

<u>Theorem</u>: Let I_p be a Riemannian metric on $\mathbb{R}^2$ such that $K=-1$ everywhere. If $(\mathbb{R}^2, I_p)$ is complete in its intrinsic metric, then $(\mathbb{R}^2, I_p)$ is isometric to $(\mathbb{H}^2, I_p)$.



and assemble pieces locally into a plane which has such a Riemannian metric

<u>What</u> about compact surfaces?

$S =$  $=$ 

Use <u>regular</u> hyp. octogon w/ vertex angles $\pi/4$. Gives nice metric on S w/ $K=-1$ everywhere.

Then $\tilde{S}$ univ cover has a Riem metric making it into $\mathbb{H}^2$! In particular, $\tilde{S} \cong \mathbb{R}^2$

$$\begin{array}{c}\tilde{S}\\ \downarrow \\ S\end{array}$$

See handout for induced tiling.

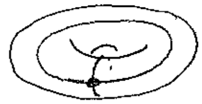
Lecture 23: Last time: Tilings of $\mathbb{H}^2$

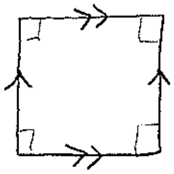
Today: Hyperbolic metrics on closed surfaces.
Back to topology: Homology.

What about compact surfaces? $\int_S K\,dA = 2\pi \underbrace{\chi(S)}$

If < 0, suggests a metric of const curve < 0.

Euclidean case:





glue sides by isometries (ok as have same length)

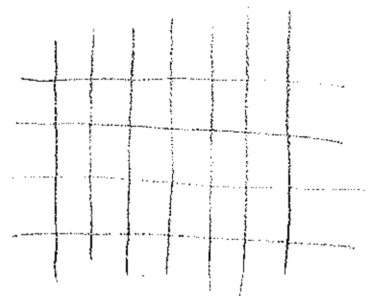
metric makes sense around vertex as angles add to 2π.

π_1  $= \mathbb{Z}^2$ of translations

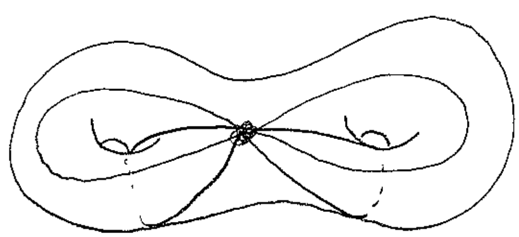
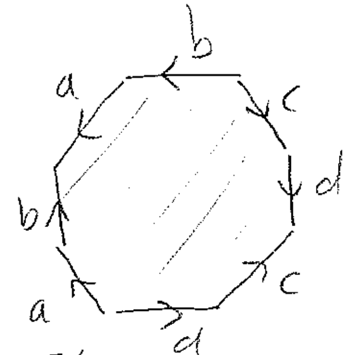
 $\xleftarrow{P} \mathbb{R}^2$

$\mathbb{R}^2/\mathbb{Z}^2$

local isometry



Hyperbolic case:

$S =$  $=$ 

Use regular octogon in $\mathbb{H}^2$ w/ vertex angles $\pi/4$.

Gives a metric on S w/ $K = -1$ everywhere.

Then $\tilde{S}$ has a Riem metric making it into $\mathbb{H}^2$ — see handout for picture.

$\tilde{S}$
$\downarrow$
S

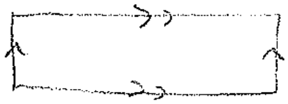
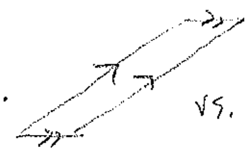
$\boxed{\text{In particular, } \tilde{S} \cong \mathbb{R}^2!}$

Works same for any surface with $X(S) < 0$.

[In general, can change any Riemannian metric into a constant curvature one:]

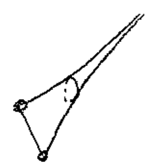
<u>Uniformization Thm</u>: Let S be a cpt surface w/ R-metric I_p. Then $\exists$ a smooth fn $\phi : S \to \mathbb{R}_{>0}$ such that $\phi(p) I_p$ is a Riemannian metric of constant curvature $+1, 0,$ or -1.

Study of constant curve metrics $\longleftrightarrow$ complex structures.

Many on T^2:  vs.  vs.  vs. 

"Moduli space of curves" $\qquad$ "Teichmüller Space"

$\mathcal{M}(S) =$ const curve metrics on S up to isom $\qquad$ $\mathcal{T}(S) =$ const curve metrics on S remembering how the surface "wears" them.

$S =$  $\qquad \mathbb{H}^2 / PSL_2\mathbb{Z} =$  $\qquad \mathbb{H}^2$

$S_g = T \# \cdots \# T$ $\qquad$ complicated $\qquad \mathbb{R}^{6g-6}$

$g \geq 2$

$\boxed{\text{Mapping class group.}}$

And now for something completely different...

Fundamental group – [easy to compute, but hard to tell answers apart.]

Measures only "1-dimensional" part of X. Can't tell S^3 from S^4!

[Need invariants that measure homology in higher dimensions.]

$\pi_1 X$ is about maps $S^1 \to X$ [Query] $S^n \longrightarrow X$ $\pi_n X$ "higher homotopy groups."

Consider skipping

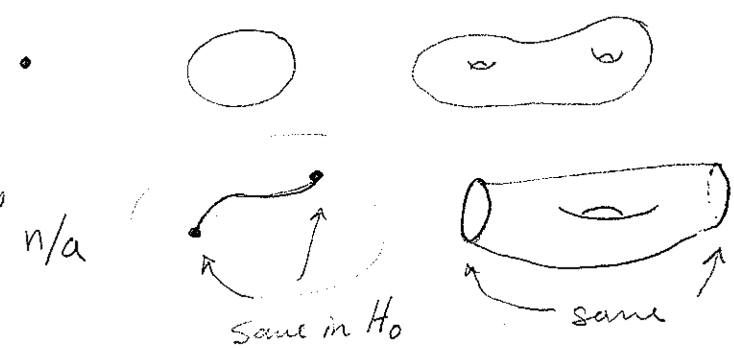
all $\pi_n S^2$ are not known

Homology: $H_n(X) = \dfrac{n \text{ dim'l things w/o boundary}}{\text{boundaries of } n+1 \text{ dim'l things}}$

$n=0$ $n=1$ $n=2$

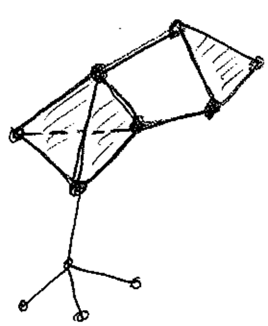
w/o ∂

boundaries of things. n/a



[Eventually, will define $H_n(X)$ for any topological space.]

K a simplicial complex [Query: finite # of simplices in $\mathbb{R}^n$]



$C_0(X) = \mathbb{Z} \oplus \mathbb{Z} = \{a_0 v_0 + a_1 v_1\}$

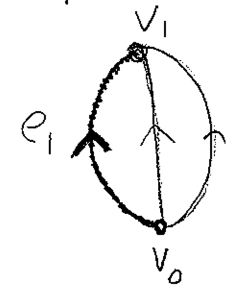
$C_1(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = \{c_1 e_1 + c_2 e_2 + c_3 e_3\}$

$C_n(X) = 0$ for $n > 1$.

Boundary map: $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ a homorphism

$\partial_1 : C_1(X) \to C_0(X)$ $\quad \partial_1(e_1) = v_1 - v_0$

$\partial_i(e_i) = v_1 - v_0$

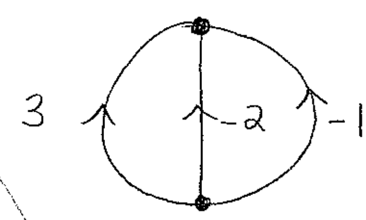


Cycle: $c \in C_n(X)$ s.t. $\partial_n c = 0$, $\ker \partial_n$

0-cycles: $\ker \partial_0 = C_0(X)$ $\quad$ [n-dim'l things w/o boundary.]

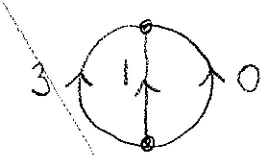
1-cycles: $\partial_1(c_1 e_1 + c_2 e_2 + c_3 e_3) = (c_1 + c_2 + c_3)(v_1 - v_0)$

$\ker \partial_1 =$ those w/ $c_1 + c_2 + c_3 = 0$



a cycle

Def: $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$



not a cycle.

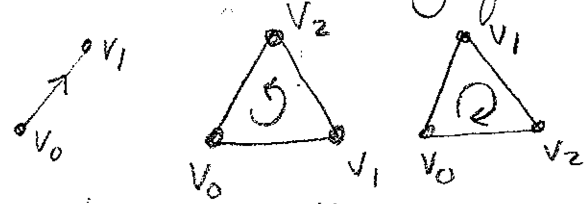
$H_0(X) = C_0(X) / \operatorname{im} \partial_1 = C_0(X) / c(v_1 - v_0)$

change of basis

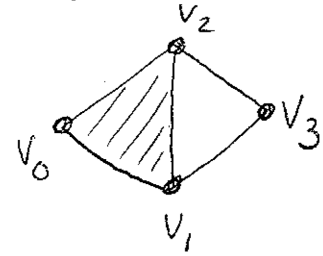
$= \mathbb{Z}^2 / \langle (1,-1) \rangle = \mathbb{Z}$

$H_1(X) = \ker \partial_1 / \operatorname{im} \partial_2 = \ker \partial_1 = \mathbb{Z}^2$ w/ basis $c_1 - c_2$, $c_2 - c_3$

Def: An *oriented simplex* is one with an ordering of the vertices $[v_0, \dots, v_n]$.



Fix an orientation on all simplices by ordering all the vertices of K. $K =$

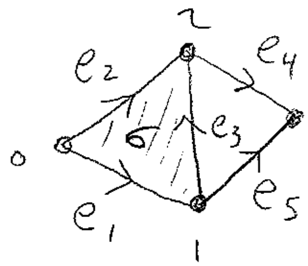


n-chains: $C_n(K) =$ free abelian gp w/ basis the n simp of K $= \bigoplus_{\sigma \text{ an } n\text{-simplex}} \mathbb{Z}$

Our ex: $C_0(K) = \mathbb{Z}^4 = \{a_0 v_0 + a_1 v_1 + a_2 v_2 + a_2 v_3\}$

$C_1(K) = \mathbb{Z}^5$: basis e_i

$e_1 = [v_0, v_1]$ etc.

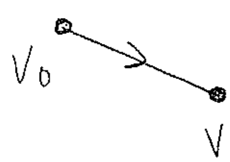


$C_2(K) = \mathbb{Z}$, basis $\sigma = [v_0, v_1, v_2]$

Boundary maps: $\partial_n : C_n(K) \to C_{n-1}(K)$ a homomorphism.

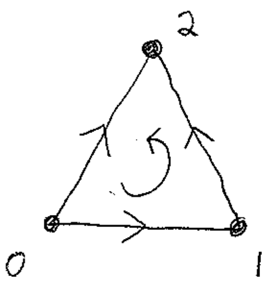
$n = 0$, $\partial_0 = 0$

$n = 1$, $\partial[w_0, w_1] = w_1 - w_0$ e.g. $\partial e_1 = v_1 - v_0$



$n = 2$ $\partial\sigma = [v_1, v_2] - [v_0, v_2] + [v_0, v_2]$

$e_3 - e_2 + e_1$



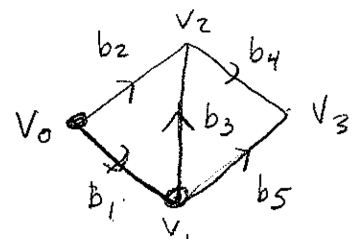
$$\partial_n([w_0, \dots, w_n]) = \sum_{i=1}^{n} (-1)^i [w_0, \dots, \hat{v_i}, \dots, v_n]$$

$$[w_0, \dots, \hat{v_i}, \dots, v_n] = [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n]$$

<u>n-cycles</u>: $\ker \partial_n$ "Things w/o boundary."

<u>Ex</u>: $n=0$ just $C_0(K)$

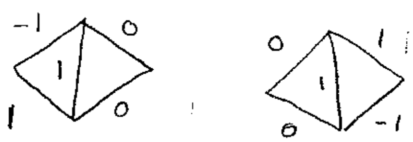
$n=1$



$$\partial(b_1 e_1 + \dots + b_5 e_5) = (-b_1 - b_2) v_0 + (b_1 - b_3 - b_5) v_1 + \dots$$

If 0 have each coor 0, i.e. $b_1 = b_3 + b_5$

Basis for $\ker \partial_n$:



z_1 $\quad$ z_2

<u>n-boundaries:</u> $\operatorname{im} \partial_{n+1}$ "boundaries of n dim'l things"

$n=0$ $\quad \operatorname{im} \partial_1 = \{ \Sigma a_i v_i \mid \Sigma a_i = 0 \}$

$n=1$ $\quad \operatorname{im} \partial_2 = z_1$

$n \geq 2$ $\quad 0$

<u>Lemma:</u> $\partial_n \circ \partial_{n+1} = 0 \implies \ker \partial_n \supseteq \operatorname{im} \partial_{n+1}$

$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}$$

<u>Thus:</u> Set $H_n(K) = \ker \partial_n / \operatorname{im} \partial_{n+1}$

In our example

$H_0(K) = \mathbb{Z}$ in general $\cong \mathbb{Z}^{\#\text{ of conn. comp}}$

$H_1(K) = \mathbb{Z}$ in general $\cong \pi_1^{ab}(|K|)$

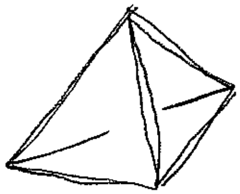
$H_2(K) = 0$

<u>Pf of lemma:</u> Suffices to check it on basis elements

$$\partial_n\left(\partial_{n+1}\left([w_0, \dots, w_{n+1}]\right)\right) = \partial_n\left(\sum_{i=0}^{n+1} (-1)^i [w_0, \dots, \hat{w}_i, \dots w_{n+1}]\right)$$

$$= \sum_{i=0}^{n+2} (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j [w_0, \dots, \hat{w}_j, \dots, \hat{w}_i, \dots, w_{n+1}] + \sum_{j=i+1}^{n+2} (-1)^{j-1} [w_0, \dots, \hat{w}_i, \dots, \hat{w}_j, \dots, w_{n+1}]\right)$$

$= 0$ as each term appears twice w/ opposite signs.

Ex 

$H_0 = \mathbb{Z}$

$H_1 = 0$

$H_2 = \mathbb{Z}$

Fact: H_n only depends on $|K|$

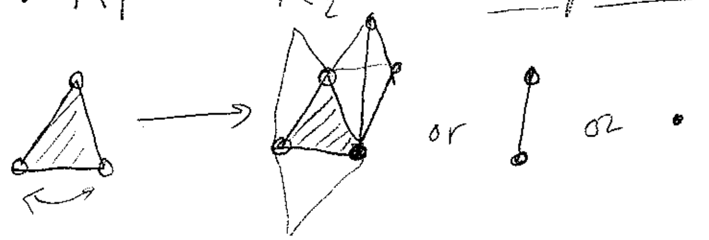
$H_n S^n = \mathbb{Z}$

Lecture 25: Last time: $H_n(K)$ for a simplicial complex.

Today: • Maps of spaces induce maps on H_n.
• Why H_n only depends on $|K|$.

$\boxed{\text{Did Not actually use}}$

$f: K_1 \longrightarrow K_2$ a <u>simplicial</u> map [Each simplex is carried onto a simplex linearly]



$f_\#: C_n(K_1) \longrightarrow C_n(K_2)$ $\qquad$ $C_n(K)$ = free abelian group w/ basis the n-simplices of K.

$$\underset{\text{an } n\text{-simplex}}{\sigma} \longrightarrow \begin{cases} \pm f(\sigma) & \text{if } f(\sigma) \text{ is an } n\text{-simplex} \\ 0 & \text{otherwise} \end{cases}$$

where the sign is the sign of the permutation:

$\sigma = [v_0, \dots, v_n]$ $\qquad$ $[w_0, \dots, w_n] \longrightarrow [f(v_0), \dots, f(v_n)]$

$f(\sigma) = [w_0, \dots, w_n]$

<u>Key</u>: $f_\#$ is a chain-map: $f_\# \circ \partial_n = \partial_n \circ f_\#$

chain complex.

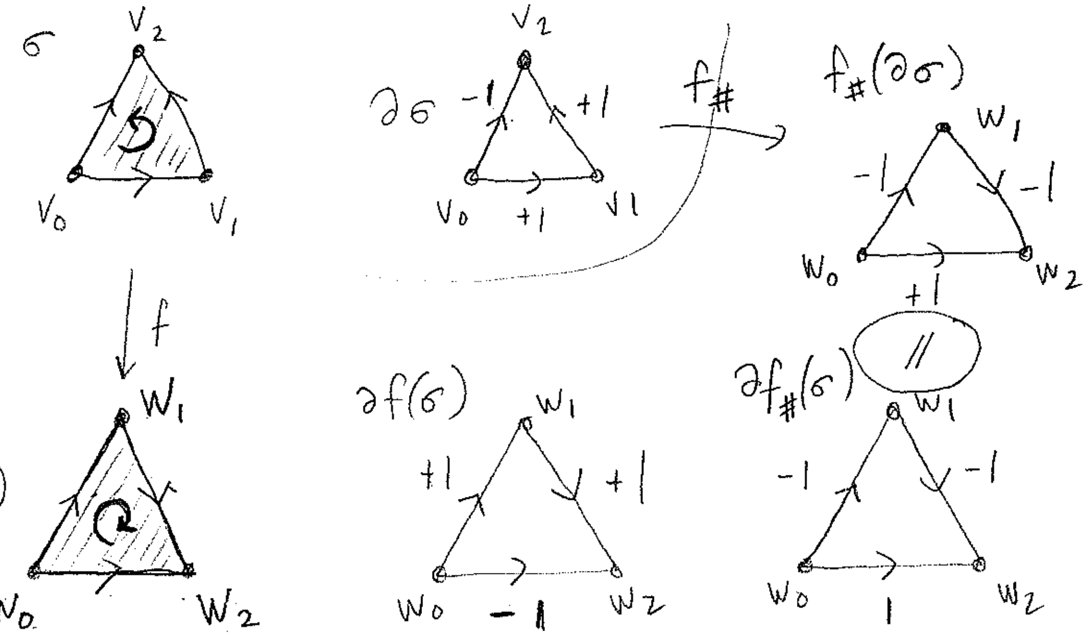
$$\begin{array}{ccccccc}
C_{n+1}(K_1) & \longrightarrow & C_n(K_1) & \xrightarrow{\partial_n} & C_{n-1}(K_1) & \longrightarrow & \\
\downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# & & \\
C_{n+1}(K_2) & \longrightarrow & C_n(K_2) & \xrightarrow{\partial_n} & C_{n-1}(K_2) & &
\end{array}$$

Consequences: $f_\#(\ker \partial_n^{K_1}) \subseteq \ker \partial_n^{K_2}$

$f_\#(\operatorname{im} \partial_{n+1}^{K_1}) \subseteq \operatorname{Im} \partial_{n+1}^{K_2}$

$$\Rightarrow \quad H_n(K_1) = \ker \partial_n^{K_1} / \operatorname{im} \partial_{n+1}^{K_1} \xrightarrow{\ f_*\ } H_n(K_2) = \ker \partial_n^{K_2} / \operatorname{im} \partial_{n+1}^{K_2}$$

Reason for key:



Didn't cover

$f_\# \sigma = -f(\sigma)$ General case is the same, breaking permutation into a product of transpositions. $(k\ k+1)$.

[see Armstrong for details.]

Thm: Suppose $f, g : K_1 \to K_2$ are homotopic maps which are simplicial. Then $f_* = g_* : H_n(K_1) \to H_n(K_2)$

Thm: Suppose $f : K_1 \to K_2$ is any map, then it is homotopic to a simplicial one on some subdivision K_1^m of K_1.

Lecture 25:

Introduced singular homology.

Gave map $H_n^{\Delta}(K) \longrightarrow H_n^{\text{singular}}(|K|)$

including the def of chain map.

Followed pages 22/23 of 2004 151a notes.

Lecture 26: Last time: Singular Homology

X top space. $C_n(X) =$ free ab. group w/ basis all $\sigma : \Delta^n \to X$

$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$ $\quad \partial_n \sigma = \sum (-1)^i \sigma \big|_{[e_0, \ldots \hat{e_i}, \ldots e_n]}$

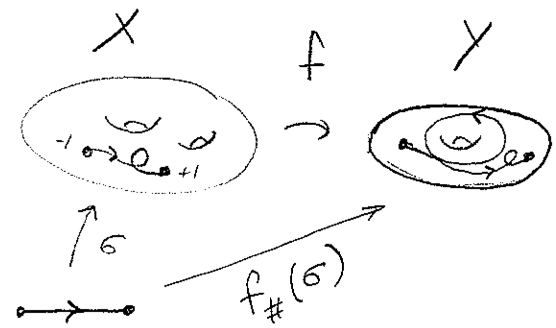
Today:

$f : X \to Y$ have $\pi_1 X \xrightarrow{f_*} \pi_1 Y$; similarly $H_n(X) \xrightarrow{f_*} H_n(Y)$

given by:

$f_\# : C_n(X) \to C_n(Y)$

$(\sigma : \Delta^n \to X) \longrightarrow \sigma \circ f$



This is a chain map, i.e. $f_\# \circ \partial_n = \partial_n \circ f_\#$

So get $H_n(X) \xrightarrow{f_*} H_n(Y)$ [Also have in simplicial homology]

Note: $X \xrightarrow{f} Y \xrightarrow{g} Z$ then $(g \circ f)_* = g_* \circ f_*$

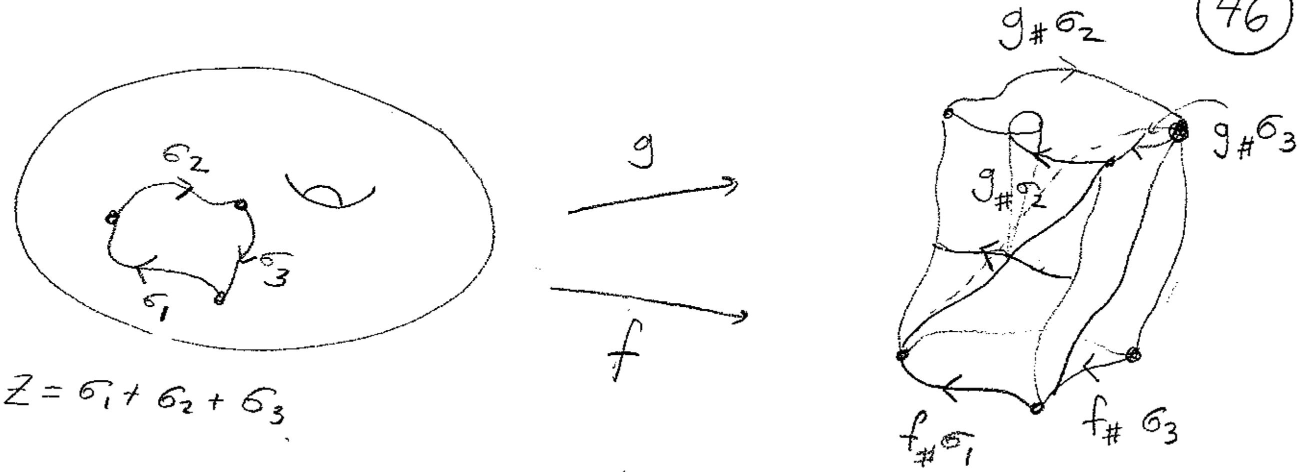
as $(g \circ f)_\#(\sigma) = g \circ f \circ \sigma = g \circ (f \circ \sigma) = f_\#(g \circ \sigma) = f_\#(g_\#(\sigma))$.

Lemma: If $f, g : X \to Y$ are homotopic maps, then

$f_* = g_* : H_n(X) \to H_n(Y)$.

Pf: See Hatcher.

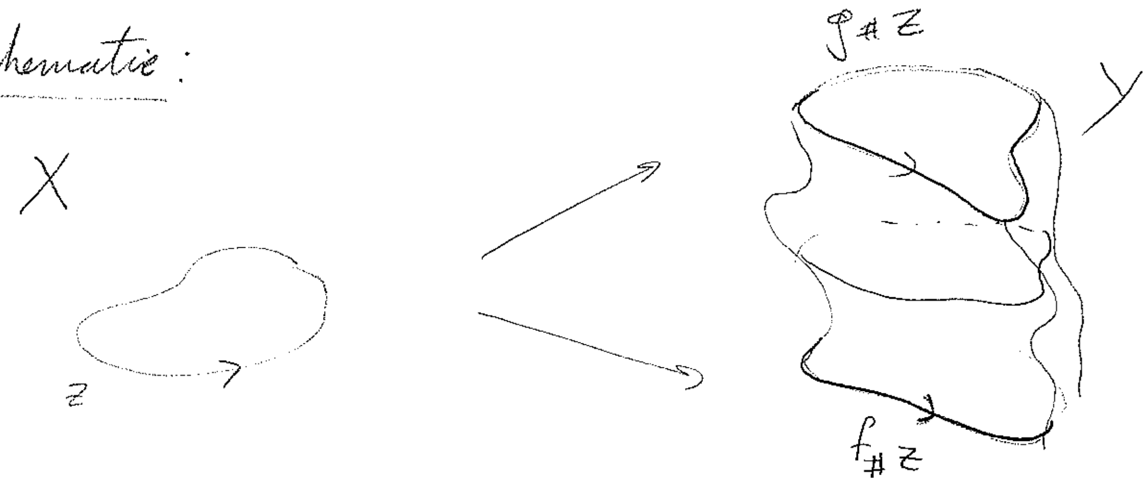
Pf idea:



$z = \sigma_1 + \sigma_2 + \sigma_3$

See that $\exists\, c$ s.t. $\partial_2 c = f_\# z - g_\# z$.

More schematic:



Thm: Suppose $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ are inverse homotopy equiv. $\left(\begin{matrix} f \circ g \simeq id_Y \\ g \circ f \simeq id_X \end{matrix} \right)$

Then $H_n(X) \xrightarrow[f_*]{} H_n(Y)$ is an isomorphism for all n.

Pf: $H_n(X) \underset{g_*}{\overset{f_*}{\rightleftarrows}} H_n(Y)$

$$g_* \circ f_* = (g \circ f)_* \overset{\text{by Lemma}}{=} (id_X)_* = id_{H_n(X)}.$$

Same for $f_* \circ g_* \implies f_*$ and g_* are inverse isom. ∎

Cor: If X is contractible, e.g. $\mathbb{R}^k$ then $H_n(X) = H_n(pt) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n>0 \end{cases}$

Lemma: $H^n(S^k = \{x \in \mathbb{R}^{k+1} \mid |x| = 1\}) = \begin{cases} \mathbb{Z} & n = 0 \text{ or } k \\ 0 & \text{otherwise.} \end{cases}$

[Saw for S^2, S^3 on HW in terms of H_n^Δ]

Key: $A^{\text{clsd}} \subseteq X$, A, X path connected, [A "reasonable"]

Can relate $H_n(A)$, $H_n(X)$, $H_n(X/A)$ via $A \xrightarrow{i} X \xrightarrow{j} X/A$

$$\to H_{n+1}(X/A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X/A) \xrightarrow{\partial} H_{n-1}(A) \to$$
$$\to H_{n-1}(X) \to H_{n-1}(X/A) \xrightarrow{\partial} \cdots \to H_1(A) \to H_1(X) \to H_1(X/A) \to 0.$$

is exact, i.e. at any term

$$U \xrightarrow{a} T \xrightarrow{b} V$$

we have $\operatorname{im}(a) = \ker(b)$. [Know two can usually deduce the 3rd]

B^n = n-dim'l ball

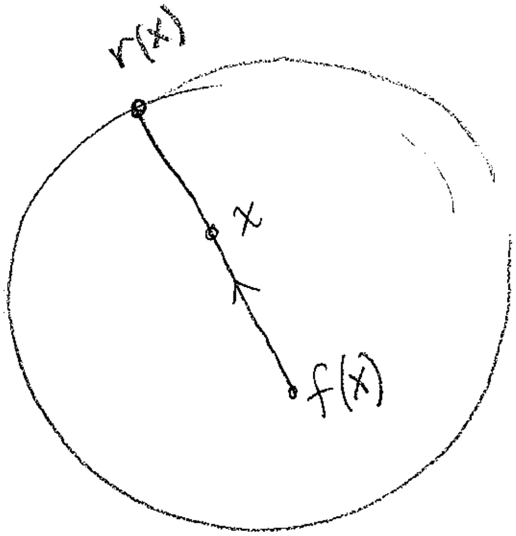
$$S^{n-1} \to B^n \to B^n/S^{n-1} = S^n$$

Cor: $\mathbb{R}^n \not\cong \mathbb{R}^m$ if $n \neq m$

Pf: $\mathbb{R}^n \setminus pt \simeq S^{n-1}$.

Thm: Any map $f: B^n \to B^n$ has a fixed pt,
i.e. an x_0 s.t. $f(x_0) = x_0$

Pf: Suppose f lacks such a fixed pt.



$$S^{n-1} \xrightarrow{i} B^n \xrightarrow{r} S^{n-1}$$

$$r: B^n \to S^{n-1} \text{ continuous}$$

$$r|_{S^{n-1}} = id \quad \text{retract of}$$

$$H_n(S^{n-1}) \xrightarrow{i_*} H_n(B^n) \xrightarrow{r_*} H_n(S^{n-1})$$

$$\boxed{id = (r \circ i)_* = r_* \circ i_* = 0}$$

Lecture 27: The final installment.

The Euler Characteristic of a simplicial complex K is $\chi(K) = \sum_{n=0}^{\infty} (-1)^n (\#\text{ of } n\text{-simplices})$

[Also makes sense for Δ-complexes w/ finitely many cells
agrees with notion for surfaces]

Thm. Let K_1, K_2 simplicial complexes. If $|K_1| \cong |K_2|$, then $\chi(K_1) = \chi(K_2)$.

Thus provided X has some triangulation, then it makes sense to talk about $\chi(X)$.

When X is a surface already have 1.75 proofs:

1) On 1st HW, via classification

[Query:] 2) Gauss-Bonnet: $\underbrace{\int_S K\,dA}_{\text{View as fixed}} = 2\pi\chi(\text{a good triangulation})$

Idea of proof will be

$$\chi(K) = \sum (-1)^n \Big(\text{rank of free part of } \underbrace{H_n(|K|)}_{\text{singular homology}}\Big)$$

$H_n(|K|) \cong \mathbb{Z} \oplus \text{finite}$

[Torsion is annoying, would be nice if homology gps are vector spaces]

Homology w/ Coefficients: F a field $[\mathbb{Z}/2$ or $\mathbb{Q}]$

$$C_n^\Delta(K) = \bigoplus_{n\text{-simplices}} \mathbb{Z} = \text{free abelian gp w/ basis } n\text{-simplices} \rightsquigarrow C_n^\Delta(K;F) = \bigoplus_{n\text{-simp}} F = \text{vector space over } F \text{ w/ basis } n\text{-simp}$$

Define boundary maps just as before, get $H_n^\Delta(K;F) = {\ker \partial_n}/{\operatorname{im} \partial_n}$ a vector space over F.

Can do same for singular homology.

Ex: $F = \mathbb{Z}/2$. [In some sense, this is simpler than orig case]

$$C_n^\Delta(K) = \bigoplus_{n\text{-simp}} \mathbb{Z}/2$$

$\Downarrow$

C is just a collection of n-simplices

There are no orientations now:

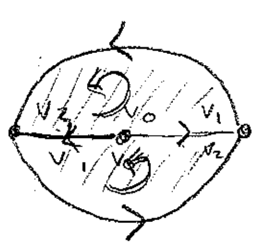
$$\partial\left(\overset{v_1}{\underset{v_0}{|}}\right) = [v_0] + [v_1]$$

$$\partial(\blacktriangle) = \triangle$$

$$H_n(S^k;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & n = 0, k \\ 0 & \text{otherwise} \end{cases}$$

For $P = [\text{proj plane}]$ we have

n \ coeff	$\mathbb{Z}$	$\mathbb{Z}/2$
>2	0	0
2	0	$\mathbb{Z}/2$
1	$\mathbb{Z}/2$	$\mathbb{Z}/2$
0	$\mathbb{Z}$	$\mathbb{Z}/2$



[Aside: S a cpt surface. Then
S contains a Möbius band $\Leftrightarrow$ $H_2(S;\mathbb{Z}) = 0$]

Pf of Thm: Suffices to show for any simp. complex

$$\chi(K) = \sum (-1)^n \dim H_n(|K|;\mathbb{Z}/2)$$

since the RHS is a top. invariant.

Consider the chain complex for $H_n^{\Delta}(K;\mathbb{Z}/2)$:

$$0 \to C_N(K;\mathbb{Z}/2) \to \cdots \to C_1(K;\mathbb{Z}/2) \to C_0(K;\mathbb{Z}/2) \to 0$$

By HW we have

$$\begin{array}{ccc} \sum (-1)^n \dim C_n(K;\mathbb{Z}/2) & = & \sum (-1)^n \dim H_n^{\Delta}(K;\mathbb{Z}/2) \\ \| & & \| \ (?!) \\ \sum (-1)^n (\# \text{ of } n\text{-cells}) & & \sum (-1)^n \dim H_n(K;\mathbb{Z}/2) \\ \| & & \\ \chi(K) & & \end{array}$$

as desired. $\blacksquare$