Ma 109b, HW #8. Due Wednesday, March 1.

Note: For the next few lectures, I will be covering material that's not in the text. For this reason, I will be scanning in my lecture notes and adding them to the course webpage.

1. Let T be the 2-torus, thought of as a smooth surface (see e.g. page 217 of the text). Show that there is a Riemannian metric on T where the Gaussian curvature K is 0 everywhere.

2. Consider hyperbolic space $\mathbb{H}^2$ in the hyperboloid model. Calculate the length of a circle $S_p(r)$ of radius r. Use this to show that the Gaussian curvature K is -1 everywhere.

   Comment: For large r, your answer should grow *exponentially* in r! This makes it very easy to get lost in hyperbolic space; in particular, a generic random walk in $\mathbb{H}^2$ moves away from the starting point at a linear rate.

3. Consider hyperbolic space $\mathbb{H}^2$, in your choice of the hyperboloid or Poincaré disc model. Consider it as a metric space with respect to the intrinsic metric.

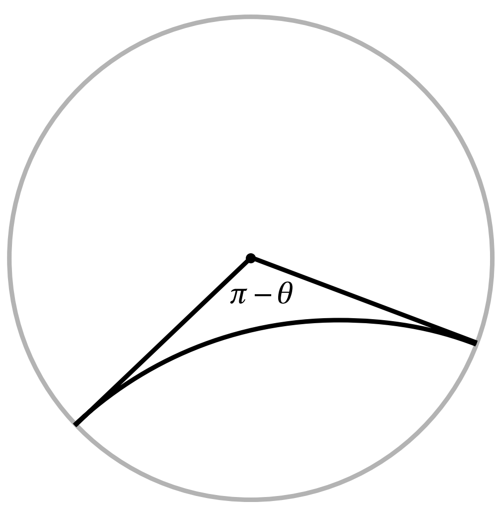
   (a) Prove that the topology induced by the metric agrees with the subspace topology (where the ambient space is $\mathbb{R}^3$ in the case of the hyperboloid model, and $\mathbb{R}^2$ in the Poincaré disc case). Hence $\mathbb{H}^2$ is simply homeomorphic to $\mathbb{R}^2$.

   (b) Prove that $\mathbb{H}^2$ with the intrinsic metric is complete, i.e. every Cauchy sequence converges.

4. In this problem, you will check directly that Gauss-Bonnet holds for triangles in $\mathbb{H}^2$. This proof goes back to Gauss.

   (a) Let $T \subset \mathbb{H}^2$ be an ideal triangle, that is, one all of whose vertices are at infinity. Check that its area is π. (This meshes with Gauss-Bonnet as the "angles" of this triangle, insofar as this makes sense, are all 0.) **Hint:** This is probably easiest in the upper-halfspace model, where the triangle has vertices $\{0, 1, \infty\}$.

   (b) Now consider a 2/3rds ideal triangle with angle $\pi - \theta$ as shown; it is unique up to isometry.



Let $A(\theta)$ be its area. Give a geometric argument that $A(\theta_1+\theta_2) = A(\theta_1) + A(\theta_2)$

(c) Use this to conclude that A is $\mathbb{Q}$-linear, i.e. $A(\frac{p}{q}\theta) = \frac{p}{q}A(\theta)$. Combining with part (a) show that $A(\theta) = \theta$ for all θ.

(d) Use part (c) to prove that if T is a triangle in $\mathbb{H}^2$ with angles $\theta_1, \theta_2, \theta_3$ then it has area $\pi - \theta_1 - \theta_2 - \theta_3$.

5. As we will see in class, it is possible to tile $\mathbb{H}^2$ with all sorts of shapes not possible in the Euclidean plane, for instance by 5-gons (right angled ones, no less), 7-gons, 10-gons, and in fact n-gons for any $n > 4$. Thus, one reasonable approximation of $\mathbb{H}^2$ is the following. Start with a bunch of regular Euclidean 7-gons, and start gluing them together along their edges, putting 3 around every vertex.

   For this exercise, build such a model out of whatever materials seem reasonable and to whatever scale you find amusing. Contrary to the usual practice, collaborators need not turn in separate "write-ups" of this problem in order to allow you to build as large a model as possible. Just put all of the collaborators names on it when you turn it in to me.

   Note: Because of the property noted in Problem 2, you will find it surprisingly difficult to make the radius large as the number of septagons needed grows exponentially. To see that you get lost easily, start on one septagon, and move to an adjacent septagon at random. Very quickly, you will step off whatever piece of hyperbolic space you made.