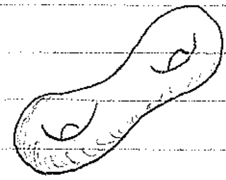
Math 157a: Lecture 1

Caltech
Winter Term
2004

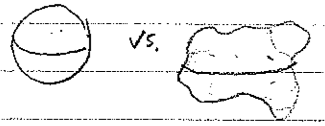
Riemannian Geometry: is about



Manifolds — Hausdorff top space locally homeo to $\mathbb{R}^n$

as w/ smooth (C^∞) structure [so can talk about vector fields, calculus]

Metric Spaces — care about the prop of particular metrics, not just the induced topology [div. line between top and geometry]



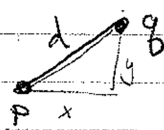
vs.

- interested in those metric spaces which remind us of Euclidean geometry.

Euclidean Plane Geometry:

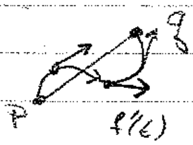
Basic concept: straight line [Secondary concepts: triangles, angles, $\parallel$ lines area, trig, etc]

$E^2 = (\mathbb{R}^2$, std Euclidean metric$)$



$d = \sqrt{x^2+y^2}$

Char: The straight line seg from p, q is the shortest path between them.



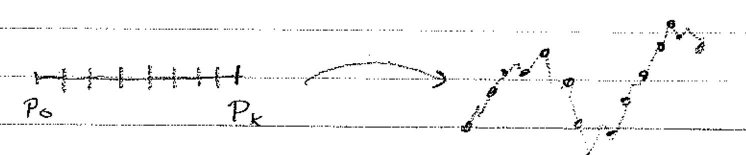
Length of path:

$f: [0,1] \to E^2, \quad f(0) = p, \ f(1) = q$

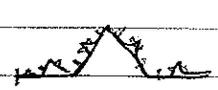
If f is C^1, $\quad \mathrm{len}(f) = \int_0^1 \|f'(t)\|\, dt.$

Gen Def: (X, d), $f: [0,1] \to X$ a cont path

Then
$$\mathrm{len}(f) = \sup_{0 = p_0 < p_1 < \cdots < p_k = 1} \sum_{i=1}^{k} d(f(p_{i-1}), f(p_i))$$



Note $\mathrm{len}(f)$ can be ∞.

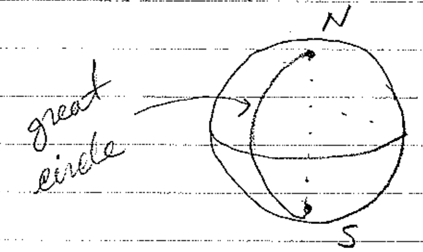


[Note sum goes up if refine the partition, by Δ ineq.]

Observe: $\text{len}(f) \geq d_X(f(0), f(1))$ by Δ-inequality

[For $\mathbb{E}^2$, can choose f = str. line so that we have $=$ here. Not always the case:]

Ex: $\left(S^2 = \{ x \in \mathbb{R}^3 \mid \|x\| = 1 \},\ \text{res of } \mathbb{E}^3 \text{ metric} \right)$



$d(N,S) = 2$

$\text{len}(\text{path}) \geq \text{len}(\text{great circ}) = \pi$

Def: (X,d) metric space. Define a new metric, the length metric, by

$$d_L(p,q) = \inf \{ \text{len}(f) \mid f \text{ is a path from } p \text{ to } q \}$$

[Comments: $d_L(p,q)$ can be ∞; (X, d_L) may not be homeo to (X,d): 

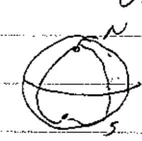
can think of as intrinsic metric ]

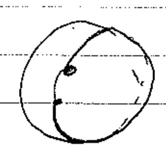
Note: $\text{len}_d f = \text{len}_{d_L} f$

[still can't always realize dist by path, e.g. $\mathbb{E}^2 - \{0\}$]

Def: (X,d) is a geodesic metric space if every two pts can be joined by a path f w/ $\text{len}(f) = d(\text{end pts})$

Length min paths are called geodesics

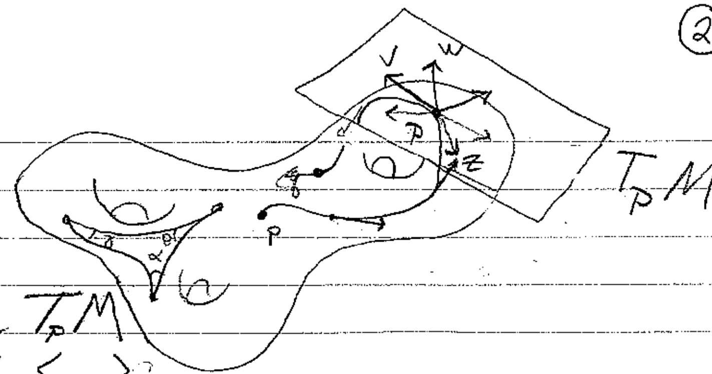
Rmk: May not be unique: 



Also: locally len min paths are usually called geodesics, too.

[A Riemannian mfld is a mfld w/ a special kind of geodesic metric space structure]

Riemannian Manifold:



M - smooth mfld

the metric $\left\{ \begin{array}{l} \text{To each tangent space } T_pM \\ \text{assign an inner product } \langle\, , \rangle_p \end{array} \right.$

$v, w, z \in T_pM$

$\alpha \in \mathbb{R}$

$\langle\, , \rangle_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ s.t. $\langle v, w\rangle_p = \langle w, v\rangle_p$, $\langle \alpha v + w, z\rangle_p = \alpha\langle v, z\rangle + \langle w, z\rangle_p$, $\langle v, v\rangle_p > 0$ for $v \neq 0$

Then $f : [0,1] \to M$ is a smooth curve set

$$\text{len}(f) = \int_0^1 \|f'(t)\|_{f(t)}\, dt \quad \text{where } \|v\|_p = \sqrt{\langle v, v\rangle_p}$$

and

$$d(p,q) = \inf \left\{ \text{len}(f) \;\middle|\; f \text{ a smooth curve joining } p \text{ to } q \right\}$$

to make M into a metric space.

Rmk: • inner prod instead of just a length so we can talk about angles.
• can replace $\langle\, , \rangle_p$ w/ other types of bilinear forms (e.g. in general rel.)
• hiding technical cond. about smoothness of $\langle\, , \rangle_p$

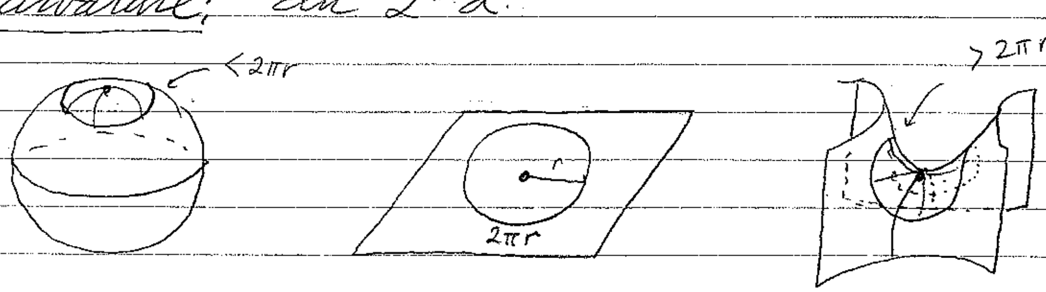
Outline of Course:

I. Background on smooth mflds: tangent spaces, tensors, diff forms

II. Basics of Riem. mflds: • defintion
• existence of geodesics
• connection, diff a vector field along a curve, normal coordinates,

[Existence of closed geod may be discussed, e.g. on S^2 w/ funny metric]

III. Curvature; In 2-d:



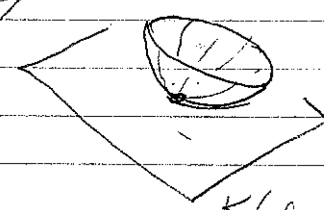
$K > 0$ $K = 0$ $K < 0$

Def: $(M, \langle , \rangle)$ a 2-d Riem. mfld. The curvature at a pt p is the # $K(p)$ such that

$$\begin{pmatrix} \text{len of circle of} \\ \text{rad } r \text{ about } p \end{pmatrix} = 2\pi r - \frac{\pi}{3} K(p) r^3 + \text{lower order terms.}$$

[normalized so that $K = +1$ for unit S^2; the reason there is no quad term is that metrics are Euclid to 3^{rd} order in normal coordinates]

In higher dimensions, have notion of sectional curvature of a plane in $T_p M$



[Interested in connections between curvature an topology]

Myers' Thm: M a cpt Riemannian mfld. Suppose every sectional curve is > 0. Then $\pi_1(M)$ is finite.

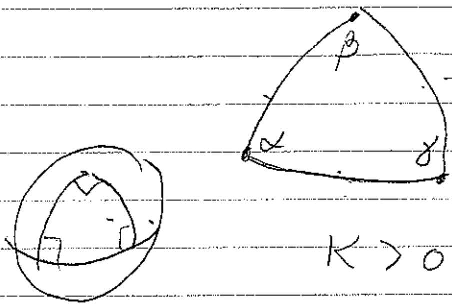
Cartan-Hadamard: M a cpt R-mfld. Suppose every sectional curvature is ≤ 0. Then $\pi_1(M)$ is infinite and $\tilde{M} \cong \mathbb{R}^n$

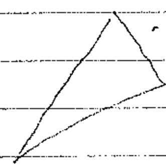
[of course, these theorems hardly form a dicotomy].

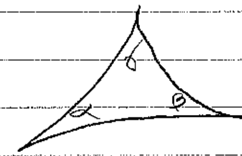
[If time permits.]

Other ways to think about curvature: Comparison Geometry

Δ w/ geod sides







$K > 0$ | $K < 0$

$\alpha + \beta + \gamma > \pi$ | $\alpha + \beta + \gamma < \pi$

Myers | Cart-Had

IV. Further topics:

[If time allows, state, say Mostow R- or something. Or geometrization.]

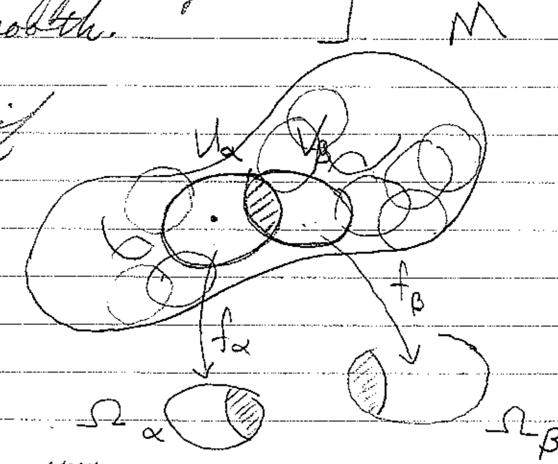
Go over syllabus, etc.

Lecture 2: Review of Smooth Manifolds

[mention Boothby as useful source, if unfamiliar]

<u>Def:</u> $f: U \to \mathbb{R}^n$, $U \overset{\text{open}}{\subseteq} \mathbb{R}^m$ is smooth (C^∞) if all partial derivatives $(\partial^k f / \partial x_{i_1} \cdots \partial x_{i_k})$ exist on all of U.

[<u>Def:</u> $f: U \to V$; U, V open $\subseteq \mathbb{R}^n$. f is a diffeomorphism if it is a bijection and f + f^{-1} are smooth.]



<u>Def:</u> A top sp M is an <u>n-mfld</u> if it is Hausdorff, has a count. basis, and every pt $p \in M$ has nbhd homeomorphic to $\mathbb{R}^n$.

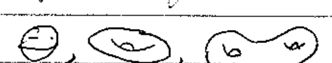
<u>Def:</u> A smooth (or C^∞) structure on a top n-mfld M is a collection

Atlas of charts $\{U_\alpha, f_\alpha\}$ where $U_\alpha \overset{\text{open}}{\subseteq} M$, $f: U_\alpha \to \Omega_\alpha \overset{\text{open}}{\subseteq} \mathbb{R}^n$ a <u>homeo</u>.

sat

- $\bigcup_\alpha U_\alpha = M$
- if $U_\alpha \cap U_\beta \neq \emptyset$ then $f_\beta \circ f_\alpha^{-1}$ is smooth

[Often require atlas to be maximal.]

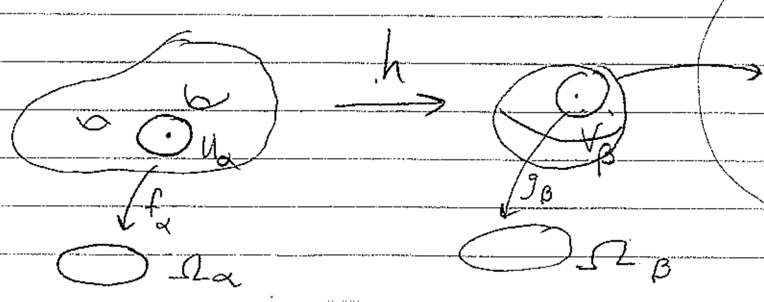


<u>Ex:</u> $\mathbb{R}^n$, Any top mfld of dim ≤ 3 has a smooth str. [higher-d, etc]

<u>Def:</u> M, N smooth manifolds. $h: M \to N$ is smooth if there are atlases $\{U_\alpha, f_\alpha\}$, $\{V_\beta, f_\beta\}$ such that $\forall \alpha, \beta$ $g_\beta \circ h \circ f_\alpha^{-1}$ is smooth.

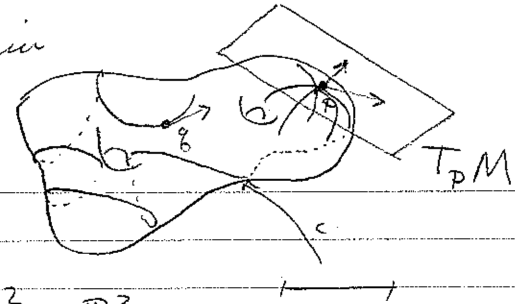
maybe omit

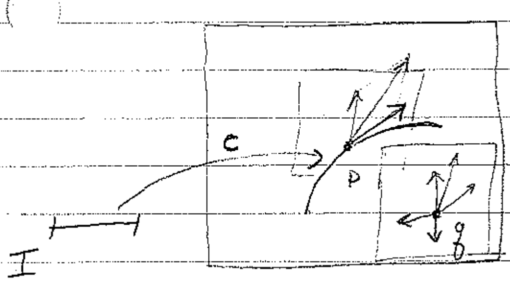
<u>Def:</u> $h: M \to N$ is a diffeomorphism if it is a bijection and h and h^{-1} are smooth



Tangent Space:

surface in $\mathbb{R}^3$





$$T_p\mathbb{R}^2 = \mathbb{R}^2$$
$$\not\parallel$$
$$T_q\mathbb{R}^2$$

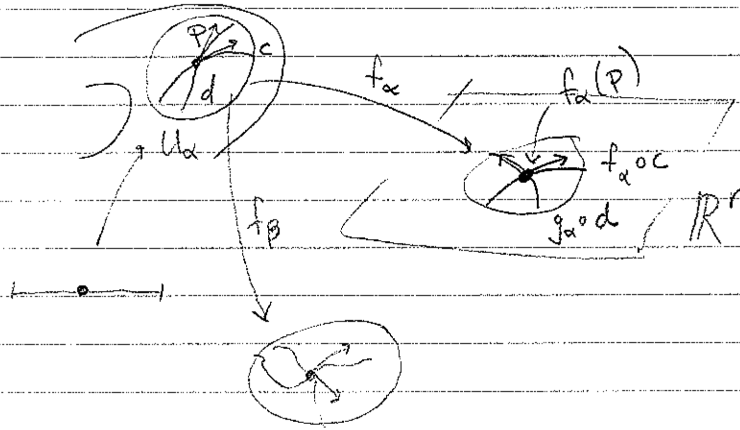
M smooth mfld, $p \in M$

1) T_pM = tangents to smooth curves through p:

A tangent vector to M at p is an equivalence class of smooth curves $c : I = [-1,1] \to M$ w/ $c(0) = p$ where

$c \sim d$ if $\exists$ a chart (U_α, f_α) around p s.t.

$$(f_\alpha \circ c)'(0) = (g_\alpha \circ d)'(0)$$



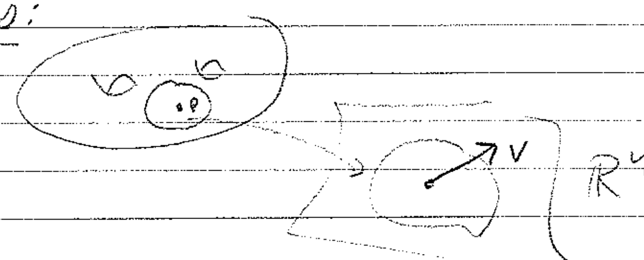
Note: Doesn't depend on chart

as
$$D = D_{f_\alpha(p)}(f_\beta \circ f_\alpha^{-1})$$
is a linear isomorphism
(as $f_\alpha \circ f_\beta^{-1}$ is also smooth!)

and $D(f_\alpha \circ c'(0)) = (f_\beta \circ c)'(0)$ etc.

2) Equivalence Classes of tangent vectors:

Set $T_q\mathbb{R}^n = \mathbb{R}^n$



Then

T_pM = equiv classes of $((U_\alpha, f_\alpha), V)$ w/ $V \in T_{f_\alpha(p)}$

where $((U_\alpha, f_\alpha), V)$ is equiv to $((V_\alpha, f_\alpha), w)$

if $\left(D_{f_\alpha(p)}(f_\beta \circ f_\alpha^{-1})\right)(\vec{V}) = \vec{w}$

[Note connection between notions, and how that makes $T_p M$ into a vector space]

[Other notion of derivation, to be discussed later]

Tangent Bundle:
$$TM = \bigcup_{p \in M} T_p M$$

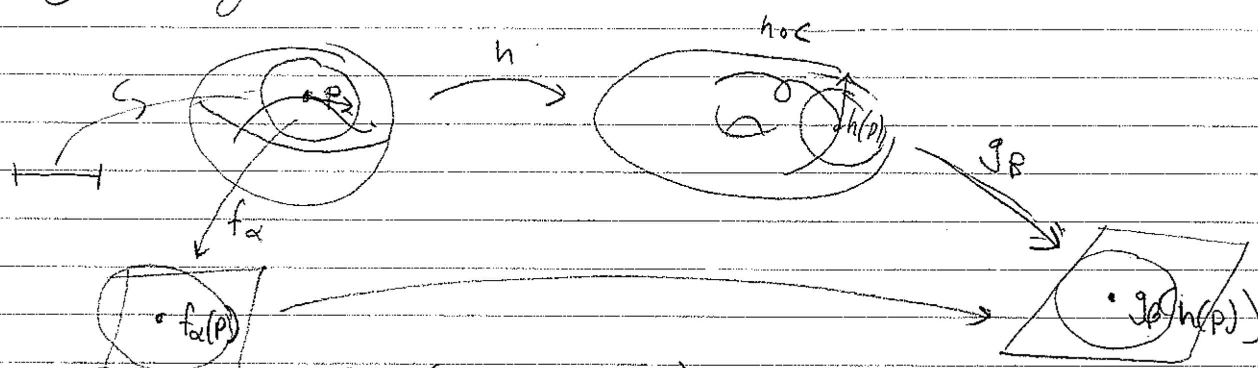
Actually, a smooth mfld:
$$T\mathbb{R}^n = \underset{(p,v)}{\mathbb{R}^n \times \mathbb{R}^n}$$, in general build up out of pieces

Ex: $TS^1 = S^1 \times \mathbb{R}$; $TS^2 \neq S^2 \times \mathbb{R}^2$ [connect to 28 diff strs on S^7]

$T\circledcirc = \circledcirc \times \mathbb{R}^2$

Because every vector field has a 0. see GHL 1.42

Derivatives: M, N smooth mflds, $h: M \to N$ smooth $p \in M$; then $\exists$ a linear map $D_p h: T_p M \to T_{f(p)} N$ given by $[c: I \to M] \longmapsto [c \circ h: I \to N]$



$$D_{f_\alpha(p)}(g_\beta \circ h \circ g_\alpha^{-1})$$

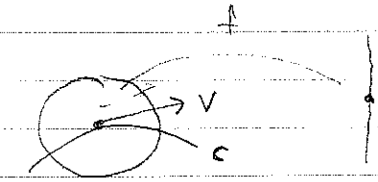
- $Dh: TM \to TM$ smooth
- have chain rule.

<u>Vector Fields</u>: assign to each pt $p \in M$ a $v_p \in T_pM$

As a fn $f: M \to TM$, must be smooth. All vect fields $\Gamma(TM)$ a linear space.

[<u>Rmk</u>: TM is trivial, $= M \times \mathbb{R}^n \Leftrightarrow \exists$ n linear indep vector fields]

Another pt of view: derivations



$v \in T_pM$, $f: U \to \mathbb{R}$

Lie derivative $L_v(f) =$ der of f in dir v $= (f \circ c)'(0)$

is a derivation:

$$L_v(fg) = f(p)\, L_v(g) + g(p)\, L_v(g)$$

Can ident $T_pM =$ all der on <u>germs</u> of fns at p.

Similarly $X \in \Gamma(TM)$ get a der $L_X: C^\infty(M) \to C^\infty(M)$

sat

$$L_X(f \cdot g) = f \cdot L_X(g) + L_X(f)\, g$$

Thm (GHL 1.51) $\Gamma(TM) \cong$ all der on $C^\infty(M)$ via $X \mapsto L_X$.

Not a derivation: $X, Y \in \Gamma(TM)$ $\quad f \mapsto X(Y(f))$

However

$$[X,Y] = XY - YX$$

is a derivation. Think of $[X,Y]$ as a vector field

[describe geometrically, if time permits]

~~Week~~ Lecture 3: [Before defining R-mfld need to talk about tensors]

Cotangent space: V v.s. $V^* = \{\text{linear } f: V \to \mathbb{R}\}$

$T_p^*M \equiv (T_pM)^* \rightsquigarrow$ cotangent bundle T^*M [again smooth]

1-form: section of $\begin{array}{c} T^*M \\ p\downarrow \;\uparrow s \\ M \end{array}$ $p\circ s$ [Example of general vector bundle]

In local coor $x_1, \ldots, x_n$ on $\mathbb{R}^n$

$T_0\mathbb{R}^n$ has basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ (or $e_1, \ldots, e_n$)

$T_0^*\mathbb{R}^n$ has basis $dx_1, \ldots, dx_n$ (dual basis)

locally 1-form is $\sum_{i=1}^{n} f(x)\,dx_i$; space of 1-forms $\Gamma(T^*M)$

[~~discuss gen vector bundle:~~ $\begin{array}{c} B \leftarrow F \\ \downarrow \\ M \end{array}$]

$T^*M \otimes T^*M$ = vec bundle w/ fiber is $T_p^*M \otimes T_p^*M \ni \alpha$

In local coor, has basis $dx_i \otimes dx_j$

is $\cong$ Bilinear fns $T_pM \times T_pM \to \mathbb{R}$

$\alpha = \sum \beta_i \otimes \gamma_i$; $\alpha(v,w) = \sum \beta_i(v)\,\gamma_i(w)$

In general, a (p,q) is a section of

$(\otimes^p TM) \otimes (\otimes^q TM)$

Will use mostly for some concrete examples,

so don't worry if seems very abstract.

Def: A Riemannian metric on [a smooth manifold M] is a smooth section g of $T^*M \otimes T^*M$ s.t. for each $p \in M$, g_p is

- symmetric: $g_p(v,w) = g_p(w,v) \quad \forall\, v,w \in T_pM$
- pos definite: $g_p(v,v) > 0 \qquad \forall\, v \in T_pM,\ v \neq 0.$

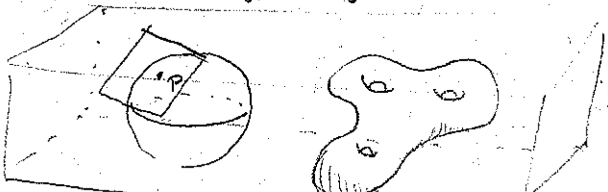
The pair (M,g) is a Riem. mfld.

Ex: $(\mathbb{R}^n$, std$)$: in coor $g = \sum_{i=1}^{n} dx_i \otimes dx_i = \sum_{i=1}^{n} dx_i^2$

Ex: $(\mathbb{R}^n$, any metric$)$: $g = \sum_{i,j} g_{ij}\, dx_i \otimes dx_j$ w/ $g_{ij} : \mathbb{R}^n \to \mathbb{R}$ smooth.

[g_{ij} sat some cond: $g_{ij} = g_{ji}$, $g_{ii} > 0$, and others : $[g_{ij}] = \underbrace{A^TA}_{\text{pos} = GL_n/O(n) \text{ so has dim } n(n+1)/2}$, $A \in GL_n\mathbb{R}$]

Ex: M a submfld of (N,g): $T_pM \subseteq T_pN$, get metric on M by restricting g

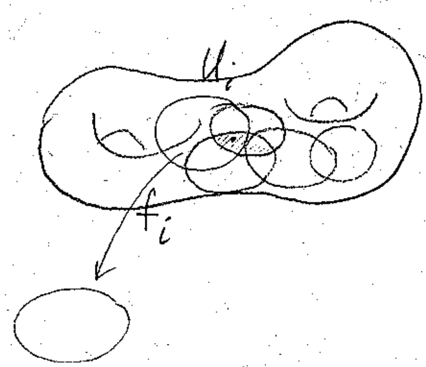


$\subseteq \mathbb{R}^3$

[case of surfaces in $\mathbb{R}^3$, studied by Gauss, is most fun much of R geom]

Ex: surfaces of revolution –

[mention other examples, perhaps at the end]



Thm: M smooth mfld. Then M has a R-metric.

Idea of Pf: $\exists$ a partition of unity (I.H):

Charts $\{U_i\}_{i=1}^{\infty}$, loc finite, smooth fns α_i supported on U_i. $\sum \alpha_i = 1$

Consider

$$g_p(v,w) = \sum_{i=1}^{\infty} \underbrace{\alpha_i(p)\, g_{std}(D_pv, D_pw)}_{\text{smooth sec of } T^*M\otimes T^*M}$$

Key: sym, pos def preserved under + in $T^*M \otimes T^*M$. ∎

(M,g) a Riem. mfld $\rightsquigarrow$ metric space structure on M

- $c:[a,b] \to M$, piecewise smooth curve $\quad L(c) = \int_a^b |c'(t)|_{c(t)}\, dt$

$$|v|_p = \sqrt{g_p(v,v)}$$

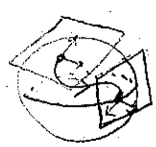
- $p, q \in M$:

$$d(p,q) = \inf\{L(c) \mid c \text{ smooth curve joining } p \text{ to } q\}$$

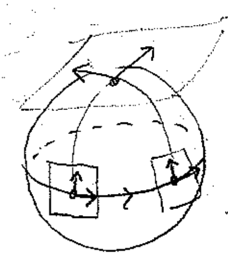
<u>Thm</u>(2.91) d is a metric on M, inducing the same top.

<u>Def</u>: (M,g) and (N,h) Riem. mflds. An <u>isometry</u> $f: M \to N$ is a diffeom s.t. $\forall p$ $D_pf: T_pM \to T_{f(p)}N$ pres. the inner product g_p and $h_{f(p)}$.

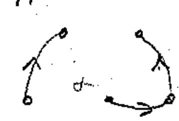
<u>Thm</u> (M,g) and (N,h) are <u>isom</u> as R-mflds iff (M, d_g) and (N, d_h) are isometric as metric spaces.
[$(\Rightarrow)$ is clear, but for $(\Leftarrow)$ see Palais P.A.M.S 1957].

<u>Parallel Transport</u>:  Want way to ident T_pM and T_qM not poss. globally $(TS^2 \neq S^2 \times \mathbb{R}^2)$

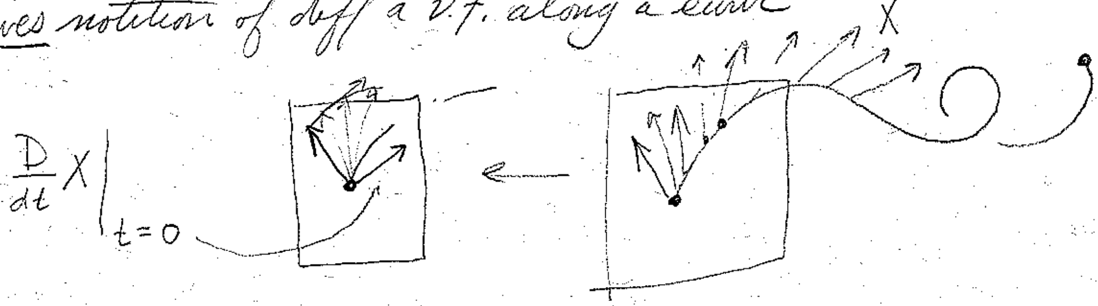
<u>Along curves</u>: • preserves inner product
• for <u>geodesic</u> c, $c'(0) \longmapsto c'(1)$.



diff between



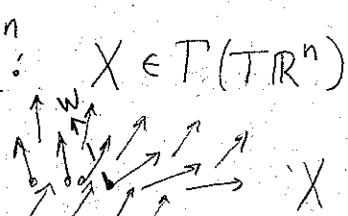
<u>Gives</u> notion of diff a v.f. along a curve



[will work "backwards": start by def diff of vector fields;
historically, parallel transport came first]

[Derivative along path is "local" so begin by discussing directional derivatives of vector fields]

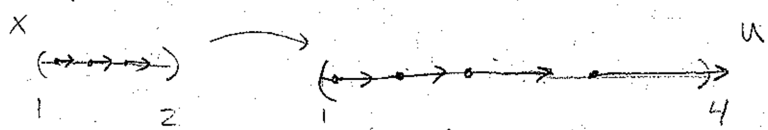
$\mathbb{R}^n$: $X \in \Gamma(T\mathbb{R}^n)$ via $T\mathbb{R}^n = \overset{(p,\ v)}{\mathbb{R}^n \times \mathbb{R}^n}$, can think of X as a fn $\mathbb{R}^n \to \mathbb{R}^n$.



$w \in T_p\mathbb{R}^n$ set $D_w X = (DX)w$ — as $\mathbb{R}^n \to \mathbb{R}^n$

Unlike dir. der of functions, doesn't extend to gen mflds as may not be preserved by change of chart maps

Ex: $f : (1,2) \to (1,4)$
$x \longmapsto x^2$
[a diffeo]



$D_w X = 0$ $\begin{cases} X = \text{const}\,\frac{\partial}{\partial x} \\ \text{on } (1,2) \end{cases} \overset{f}{\Longrightarrow} \begin{cases} X' = 2\sqrt{u}\,\frac{\partial}{\partial u} \\ \text{on } (1,4) \end{cases}$ but $D_{(2,\frac{\partial}{\partial u})} X' = \frac{1}{\sqrt{2}}$

Lecture 4:

Prop of $D_w X$ on $\mathbb{R}^n$:
$$\left.\begin{aligned} D_{\alpha v + w} X &= \alpha D_v X + D_w X \quad \alpha \in \mathbb{R} \\ D_v(\alpha X + Y) &= \alpha D_v X + D_w(Y) \end{aligned}\right\} \text{bilinear}$$

For $f \in C^\infty(\mathbb{R}^n)$, $D_v(fX) = D(fX)\cdot v = (D_v f)X + f D_v X$

If $Y \in \Gamma(T\mathbb{R}^n)$, can talk about $D_Y X \in \Gamma(T\mathbb{R}^n)$ by $(D_Y X)_p = D_{Y_p} X$. Then $D_{fY} X = f D_Y X$

Then $D_X Y - D_Y X = [X, Y]$ (symmetry)

[why is plausible: $D_X Y$ measures twisting of Y w.r.t to X, so difference measures non-comm. of X, Y e.g. $[X, Y]$]

Pf of symm: 1) compute in local coordinates.

2) Consider $T = D_X Y - D_Y X - [X, Y] : \Gamma(T\mathbb{R}^n) \times \Gamma(T\mathbb{R}^n) \to \Gamma(T\mathbb{R}^n)$

Claim: T is a tensor, i.e. $T(X,Y)_p$ dep only on X_p and Y_p.

so can think of it as $\in \Gamma(T^*\mathbb{R}^n \otimes T^*\mathbb{R}^n \otimes T\mathbb{R}^n = \mathrm{Hom}(T\mathbb{R}^n \otimes T\mathbb{R}^n, T\mathbb{R}^n)$

Why? T is $C^\infty(M)$-linear in each var, that is

$$T(fX + Y, Z) = f\,T(X, Z) + T(Y, Z) \qquad f \in C^\infty(M),\ X, Y, Z \in \Gamma(T\mathbb{R}^n)$$

same for other var. By GHL 1.114, T is a tensor.

$$T(fX, Y) = D_{fX} Y - D_Y fX - \underbrace{[fX, Y]}_{(fX)Y - Y(fX)}$$

$$= f D_X Y - \cancel{(Yf)} X - f D_Y X - (f[X,Y] - \cancel{(Yf)} X)$$

$$= f\,T(X,Y)$$ [since all terms are 0.]

So T is a tensor. So what? Well note $T(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = 0 \quad \wedge$

As T is a tensor, this implies $T = 0$.

Def: A connection on M is a $\mathbb{R}$-bilinear map

$D : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$ satisfying

i) $X, Y \in \Gamma(TM)$, $f \in C^\infty(M)$ then

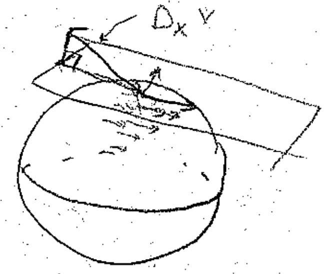
$$D_{fX} Y = f D_X Y \quad \text{and} \quad D_X(fY) = (Xf)Y + f D_X Y$$

ii) $$D_X Y - D_Y X = [X, Y]$$

Notes: many use ∇ instead of D

for some, this is a symmetric connection, and a connection need only satisfy i)

$\mathcal{E}x$: Motivating example: $\mathbb{R}^n$, std D.



$\mathcal{E}x$: M an embedded surface in $\mathbb{R}^3$.
connection D on M as follows
$X, Y \in \Gamma(TM)$, choose extensions $\tilde{X}, \tilde{Y}$ to open set $U \supseteq M$. Then

$$D_X Y = \text{orthogonal projection of } D^{\mathbb{R}^3}_{\tilde{X}} \tilde{Y} \text{ onto } T_p M$$

[Doesn't depend on $\tilde{X}, \tilde{Y}$ as $D_{\tilde{X}} \tilde{Y}$ only dep on $\tilde{X}_p$, and only need to know $\tilde{Y}$ along some curve w/ tangent $\tilde{X}_p$]

$\mathbb{R}$-bilinearity and i.) follow immediately as proj_{T_pM} is linear
ii) need: $\underbrace{[X,Y]_p}_{\text{taken in } M} = \underbrace{[\tilde{X},\tilde{Y}]_p}_{\text{in } \mathbb{R}^3}$ for $p \in M$ which is clear from flow pt of view.

[★ Gives examples on any orientable surface.]

[Want connections which respect the R-metric so that induced parallel transport preserves the inner product]

Thm (M,g) a R-mfld. Then $\exists$ a unique connection D s.t. $\forall X, Y, Z \in \Gamma(TM)$:

$$X g(Y,Z) = g(D_X Y, Z) + g(Y, D_X Z)$$

$\mathcal{E}x$: $\mathbb{R}^n$, D_{std} this is just the product rule. $Y = (Y^1, \ldots, Y^n)$, $Z = (Z^1, \ldots, Z^n)$.

$$X \sum Y^i Z^i = \sum X(Y^i) Z^i + \sum Y^i X(Z_i) = D_X Y \cdot Z + Y \cdot D_X Z.$$

$\mathcal{E}x$: $M^2 \subseteq \mathbb{R}^3$ w/ connection above.

Def: The connection is called the Levy-Civita (or Riemannian) connection on (M,g).

Pf: First uniqueness: Let D be such a connection. Then

$$X g(Y,Z) - g(D_X Y, Z) - g(Y, D_X Z) = 0$$

$$Y g(Z,X) - g(D_Y Z, X) - g(Z, D_Y X) = 0$$

$$-Z g(X,Y) + g(D_Z X, Y) + g(X, D_Z Y) = 0$$

$$0 = X g(Y,Z) + Y g(Z,X) - Z g(X,Y) - g(Z, D_X Y + D_Y X) \qquad = D_Y X - D_X Y + 2 D_X Y$$

$$+ g(Y, \underbrace{D_Z X - D_X Z}_{[Z,X]}) + g(X, \underbrace{D_Z Y - D_Y Z}_{[Z,Y]})$$

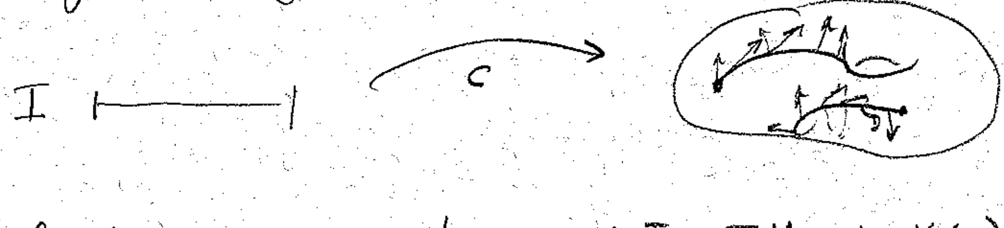
So

$$2 g(Z, D_X Y) = X g(Y,Z) + Y g(X,Z) - Z g(Y,X) + g(Z,[X,Y]) + g(Y,[Z,X]) + g(X,[Z,Y]) \qquad (*)$$

If X, Y are fixed, choosing Z makes determine $D_Y X$.

For existence, use formula $(*)$, check is a connection. First check is C^∞ linear in Z. So RHS of $(*)$ only depends on Z_p, for fixed X, Y. Check i) and ii) ∎

Vector fields along curves:



c curve

A v.f. along c is a ~~smooth~~ map $X: I \to TM$ s.t. $X(t) \in T_{c(t)} M$

<u>Thm</u>: (M, g) Riem mfld w/ conn D, $c: I \to M$ a smooth curve

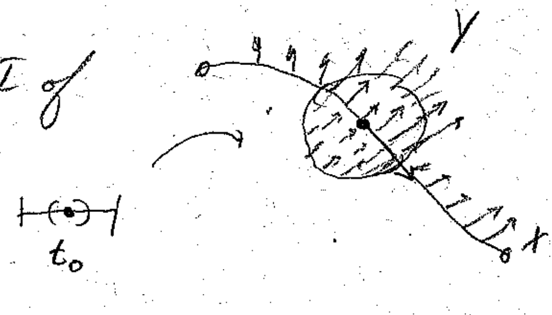
$\exists!$ linear operator $\frac{D}{dt}$ defined on vector fields along c satisfying

i) $f \in C^\infty(I)$ then

$$\frac{D}{dt}(fX)(t) = f'(t) X(t) + f(t) \frac{D}{dt} X(t)$$

ii) if X is the restriction on an open nbd of $t_0 \in I$ of a v.f. Y on M then

$$\frac{D}{dt} X(t_0) = D_{c'(t_0)} Y$$

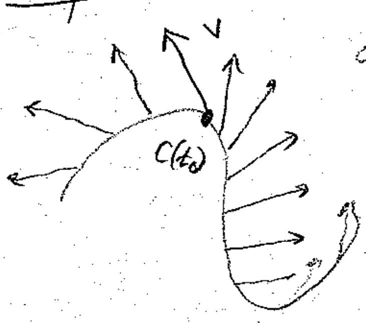


<u>Pf</u>: Essentially use ii) <u>locally</u> to define $\frac{D}{dt}$. Only issue is that extention may not be unique. See GHL 2.68.

<u>Parallel Transport:</u>

<u>Def</u>: A v.f. X along a curve c is parallel if $\frac{D}{dt} X = 0$.

<u>Prop</u>: $c: I \to M$ curve, $v \in T_{c(t_0)} M$. $\exists!$ parallel v.f X along c s.t. $X(t_0) = v$.



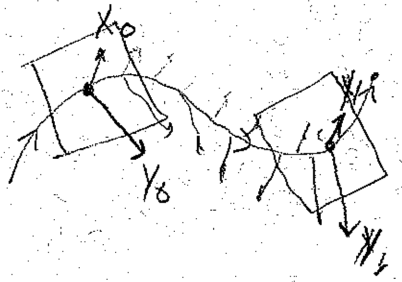
<u>Pf</u>: Existence and uniqueness of solutions to systems of 1^{st} order ODE's.

[See e.g. Boothby appendix to chapter IV, for ODE theorem]

Lecture 5:

Def: c curve in M. define parallel transport $T_{c(t_0)}M \longrightarrow T_{c(t_1)}M$

by $v \longmapsto X(t_1)$ where X is a $\parallel$ v.f. w/ $X(t_0) = v$.



Prop: $\parallel$ transport is a linear isom which preserves the inner prod

Pf: linear: [clear from prop of $\frac{D}{dt}$]

isom: [clear from reversing time]

preserves inner product: Observe X, Y are v.f. along c

$$\frac{d}{dt} g(X(t), Y(t)) = g\left(\frac{D}{dt}X(t), Y(t)\right) + g\left(X(t), \frac{D}{dt}Y(t)\right)$$

so if X, Y are $\parallel$, then $\frac{d}{dt} g(X,Y) = 0$ so $g(X,Y) = $ const

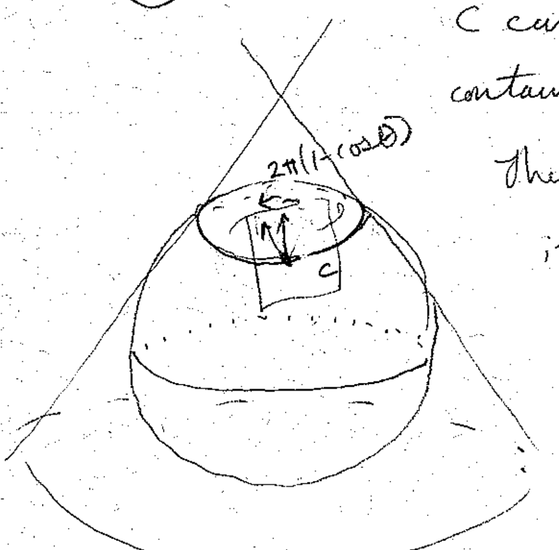
Ex: $\mathbb{R}^n$: $\parallel$ transport does not depend on path
[will char "flatness" locally]

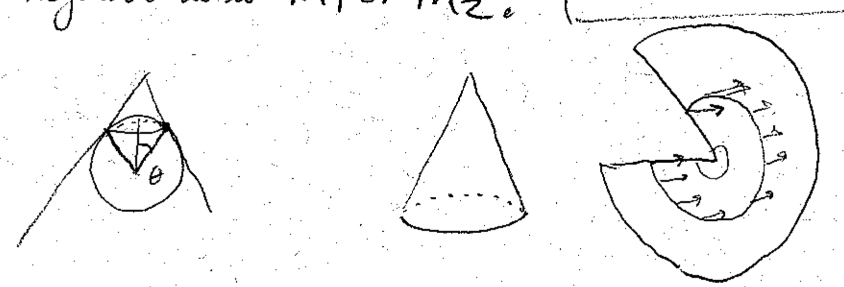
Ex:  $\subseteq \mathbb{R}^3$. Connection = std + projection

c curve M_1. Suppose M_2 is a second surface containing c s.t. $TM_1 = TM_2$ along c

Then $\parallel$ trans. along c is the same whether it is regarded as in M_1 or M_2.

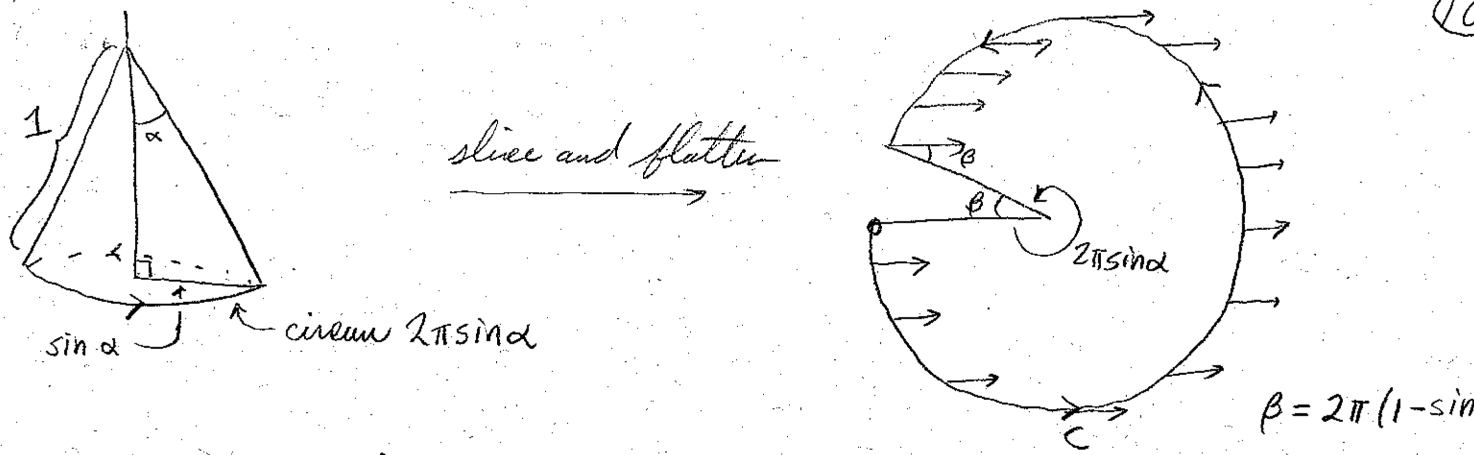
$\boxed{\text{rotation by } 2\pi(1-\cos\theta)}$



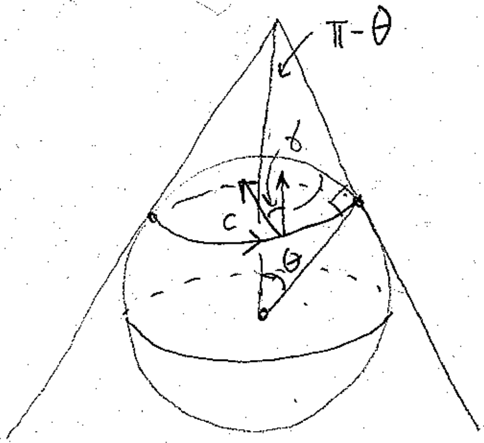


in $\mathbb{R}^2$

Generalizes to surfaces of rotation.



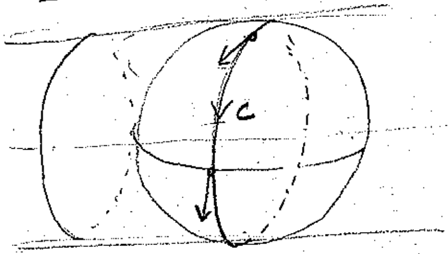
$\beta = 2\pi(1-\sin\alpha)$



$\gamma = 2\pi(1-\cos\theta)$

Generalizes to surfaces of revolution

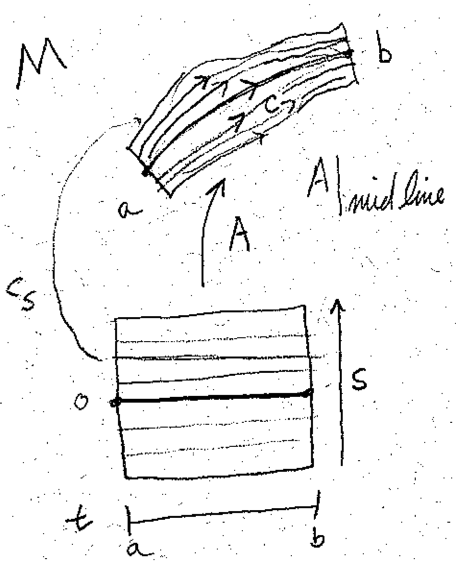
Geodesics: [locally lenth minimizing paths; mention what I said earlier about ∥ trans along geod]



If c is param. by arc length, then

$\frac{D}{dt} c' = 0.$ [Now annother way]

Euler-Lagrange Equation [first var. formula, could omit as on the HW?]



Suppose c param. by unit speed.

Then

$$\frac{d}{ds} L(c_s) = \langle c', V\rangle\Big|_a^b - \int_a^b \langle V, \tfrac{D}{dt} c'\rangle\, dt$$

where

V is the vector field $DA\left(\frac{\partial}{\partial s}\right)$ along c.

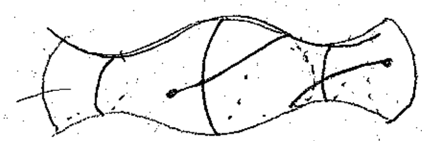
Thus if c is shortest path, $\frac{D}{dt} c' = 0$.

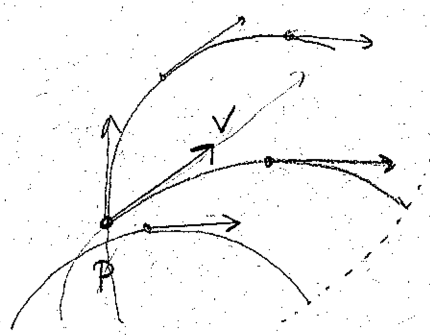
[For now, lets just define as follows; owe proof that such guys are locally length minimizing]

Def: A smooth curve c in a R-mfld (M, g) is a <u>geodesic</u> if $\frac{D}{dt} c' = 0$

Geod are necess. const speed: $\frac{d}{dt}\langle c', c'\rangle = 2\langle \frac{D}{dc} c', c'\rangle = 0.$

Examples: [$\mathbb{R}^n$: straight lines, great circles on S^2]

Flat tori:  HW: 



[Given $p \in M$, $v \in T_pM$ does there exist a geod starting at p w/ initial velocity v.]

Thm $p \in M$ a R-mfld. Then $\exists$ a nbhd U of p and $\varepsilon > 0$ s.t.

$\forall p_0 \in U$, $v \in T_{p_0}U$ w/ $\|v\| < \varepsilon$, $\exists$ a unique geodesic $c_v : (-1, 1) \to M$

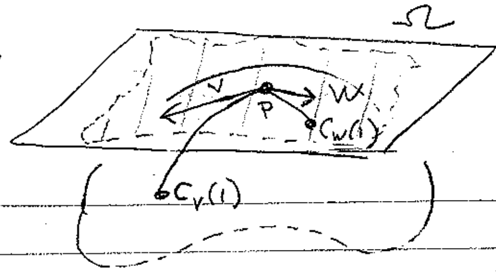
w/ $c_v(0) = p_0$ and $c_v'(0) = v$.

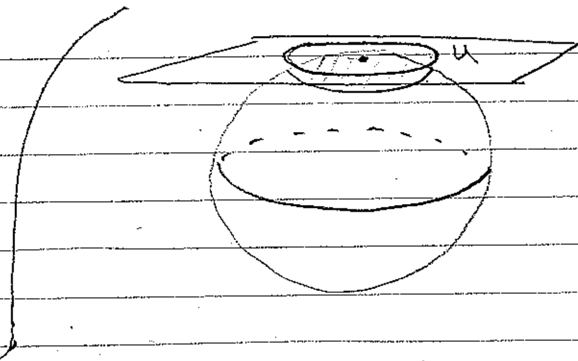
Set $W = \{ v_p \in TU \mid \|v_p\| < \varepsilon \}$. Then the map $C : W \times (-1,1) \to M$

given by $(v_p, t) \to c_{v_p}(t)$ is smooth.

Pf: Write in local coordinates, see that this gives a 2^{nd} order ODE.
Use exist, unique, smooth dep on initial conditions (GLH 2.84–2.85) ■

Exponential Map:





$$\Omega = \{ V_p \in TM \mid C_{V_p}(1) \text{ is defined} \}$$

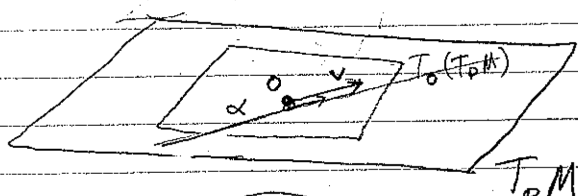
$\exp : \Omega \to M$ via $V_p \longmapsto C_{V_p}(1)$

exp is smooth by last thm.

Thm: For $p \in M$, $\exists$ a nbhd U of $0 \in T_pM$ s.t. exp is a diffeo $U \to$ (nbhd of p in M)

Pf: Restrict exp to T_pM; by Inverse Function Theorem enough to check that $D_0 \exp$ is invertable. In fact,

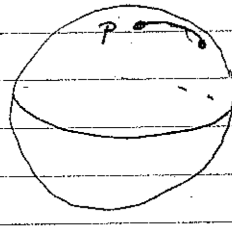
$D_0 \exp : T_0(T_pM) \to T_pM$ is the ident if we identify $T_0(T_pM)$ with T_pM: $V \in T_0(T_pM)$ can be rep by the curve $\alpha(t) = tV$



Now $\exp \circ \alpha(t) = C_{tV}(1) = C_V(t)$

Thus $D_0 \exp(V) = (\exp \circ \alpha)'(0) = C_V'(0) = V$

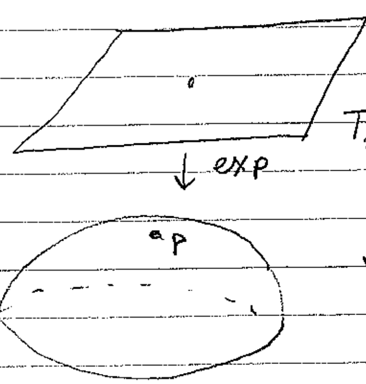
and we're done. ∎



[* connect, uniqueness of geod in ball, see GHL 2.88]

Uses of Exp:
- normal coordinates
- global param, when is exp a covering map
- understand curve in higher dim

Gauss Lemma:



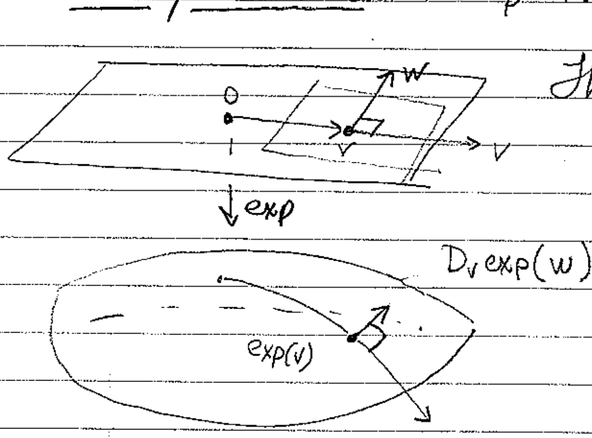
T_pM - R-mfld w/ metric g_p [isometric to $(\mathbb{R}^n, \text{std})$]

To what extent is $\exp : T_pM \to M$ an isometry?

Is at 0, not where unless M is flat, e.g. a flat torus.

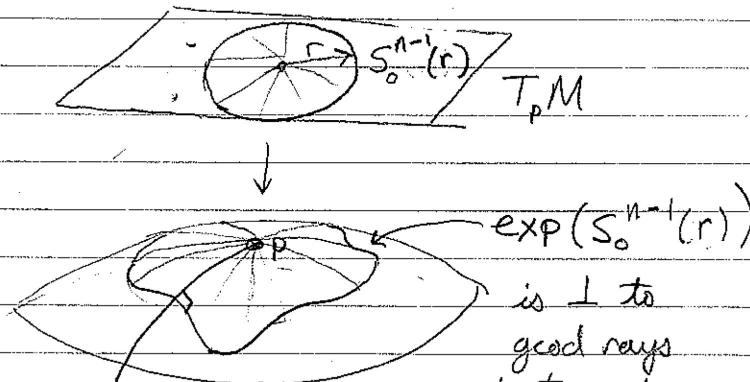
[purpose: to show local uniqueness of geodesics, etc]

Gauß Lemma: $v \in T_pM$. Let $w \in T_v(T_pM)$ which is $\perp$ to v.



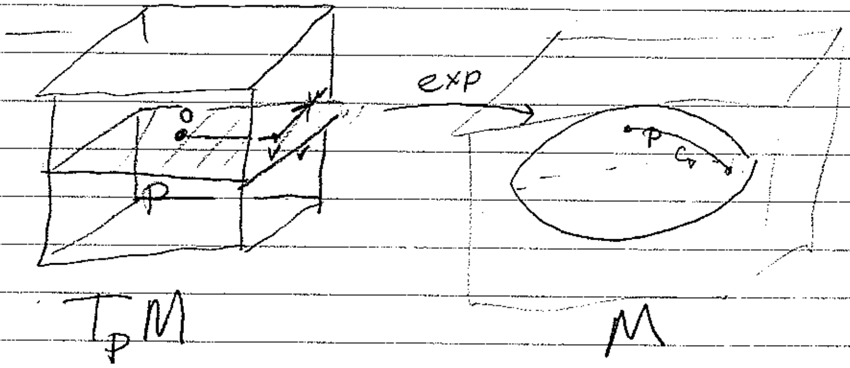
Then $D_v\exp(v)$ and $D_v\exp(w)$ are $\perp$ in $T_{\exp(v)}M$

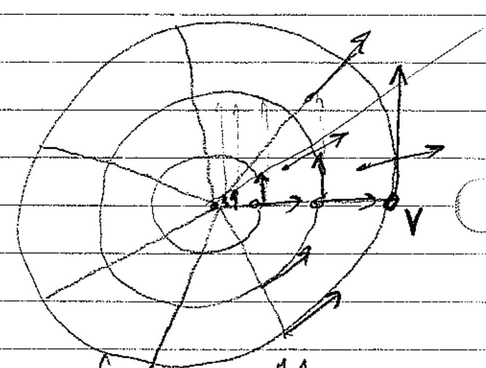
Geom restatement:



[Could give plausibility argument on why angle should
be right. shortens, etc.]

Proof: To simplify assume $B = B_0(2|v|)$ is mapped by exp diffeo into M via $\exp: T_pM \to M$. Let $P \subseteq T_pM$ be the plane containing $0, v, w$. By looking at $\exp|_P$ can reduce to the 2-dim case.





Now consider polar coor (r,θ) in T_pM, and consider v.f. $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$. Let R, T to be the images of these v.f. in M.

Want $g_{\exp(v)}(R,T)=0$. On a punctured nbhd of p, we have.

$$Rg(R,T) = g(\overset{\overset{0 \text{ as is a geodesic}}{\parallel}}{D_R R}, T) + g(R, D_R T)$$
$$= g(R, D_T R) \qquad \text{as } [R,T]=0.$$
$$= \tfrac{1}{2} T g(R,R) = 0$$

Thus $g(R,T) = $ a const C on (Nbhd of p) $\setminus \{p\}$

Consider
$$A(r) = g_{\exp(tv)}\left(R, \tfrac{1}{r}T\right)$$

Then
$$A(r) = \tfrac{1}{r} C \quad \text{for } r>0.$$

As $\lim_{r\to 0} A(r)$ exists, must have $C=0$. Thus $g_{\exp p(v)}(R,T)=0$ as required. [put up Gauss Lemma before class] $\blacksquare$

Lecture 6

Thm: Let $p \in M$. Let $B_0(r) \subseteq T_pM$ be such that $\exp_p$ is an embedding. Then

(i) For $v \in B_0(r)$, $C(t) = \exp_p(tv)$ is the unique curve in M w/

$$L(c) = d_g(p, c(1)) = \|v\|_p$$

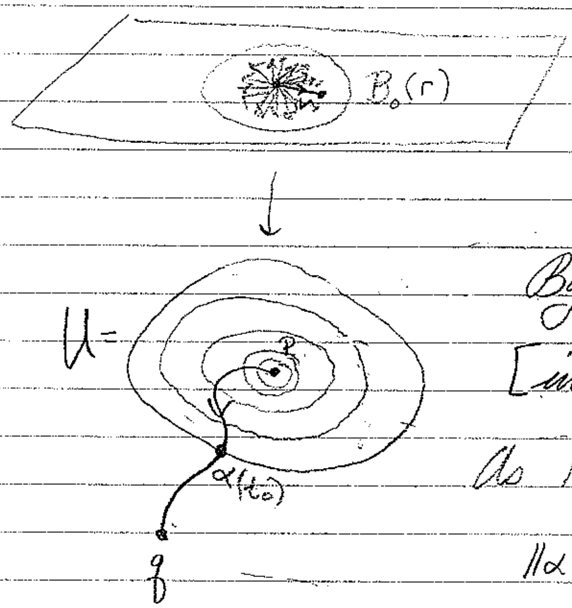
In particular if α is any curve in M joining p to $c(1)$ w/ $L(c) =$ dist, then α is a reparam of c.

(ii) If $q \notin \exp(B_0(r))$ then $d(p,q) \geq r$.

[Work in heuristic description of pf somewhere: measuring prog. through rad spheres, etc, and gauss lemma]

$\alpha : [0,b] \to M$

Pf: ii) Let α be a piecewise smooth curve joining p to $q \notin \exp(B_0(r))$. Must show $L(\alpha) \geq r$. Let t_0 be the least t s.t. $\alpha(t) \notin \exp(B_0(r))$. Thus $\alpha|_{[0,t_0)} \subseteq \exp(B_0(r))$



Let $U = \exp(B_0(r))$ and set $f : U \to \mathbb{R}$ be the push forward of radial dist fn on $B_0(r) \subseteq T_pM$.

By Gauss Lemma, $\operatorname{grad}(f) = \exp_*\left(\frac{\partial}{\partial r}\right) = R$

[insert aside about gradient].

As $\|R\|_p = 1$, we have

$\|\alpha'(t)\| \geq g(R, \alpha'(t))$ where α' is defined $t \in (0, t_0)$

Thus

$$L(\alpha) \geq L(\alpha|_{(0,t_0)}) = \int_0^{t_0} \|\alpha'\|\, dt \overset{*}{\geq} \int_0^{t_0} g(R, \alpha'(t))\, dt$$

$$= \int_0^{t_0} df(\alpha')\, dt = f(t_0) - f(0) = r.$$

i) That c_v has min length follows from ii) w/ $r = \|v\|$.

Thus $d(p,q) = \|v\|$. For uniqueness note ii) implies that any other length min curve α is contained in $\exp(\overline{B}_0(\|v\|))$

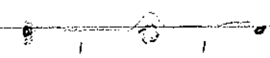
Moreover must have $(*)$ be equality, and thus $\alpha'(t)$ is a linear mult of R for all R. Thus α is a reparam of c_v as required. ∎

Cor: Length min paths are geodesics

Pf: 

So geod are locally length min. [not always globally]

Points need not be jointed by geod ($\mathbb{R}^2 \setminus \{0\}$, std)

Prob: geod don't ~~extend~~ extend for all time 

Hopf-Rinow: The following are equivalent for (M, g) $d_g = 2$

a) (M, d_g) is complete as a metric space.

b) $\exists p \in M$ s.t. exp is defined on all of T_pM

c) exp is def on T_pM for all p

Any of these implies

d) any two pts are joined by a minimal geod c w/ $L(c) = d_g(p, q)$

[Most common use a) $\Rightarrow$ rest esp d). a) often holds e.g.:]

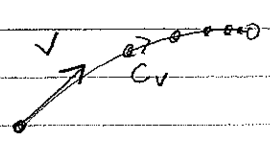
Cor. Suppose M is cpt. Then exp is def on all of TM and any two pts are joined by a min geodesic.

[Ask why d) does not imply a), e.g. open disk $\subseteq (\mathbb{R}^2, \text{std})$]

Pf: a) $\Rightarrow$ c) Let $p \in M$, $v \in T_pM$ w/ $\|v\|_p = 1$.

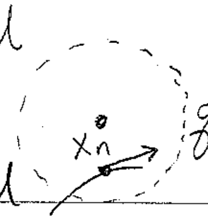
Let c_v be the cor. unit speed geod, defined on a maximal int $[0, t_0)$. Set $x_n = c_v(t_0 - 1/n)$. Then

$$d_g(x_n, x_m) \leq 1/\min(n, m)$$



as x_n and x_m are joined by a path (c_v!) of len $\leq 1/\min(n,m)$. So $\{x_n\}$ is Cauchy. Let q be $\lim(x_n)$.

Then $\exists$ nbhd U of q and an $\varepsilon > 0$ s.t. any unit vector w in TU has cor geod of length $> \varepsilon$ through it.



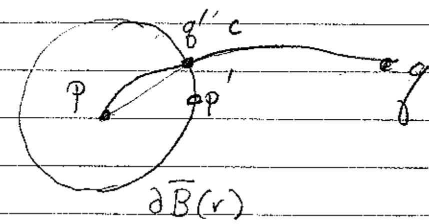
Choose n s.t. $x_n \in U$ and $1/n < \varepsilon$. Then c_v can be ext for at least time ε past $t_0 - 1/n$, contradicting max of t_0. Thus c_v is defined for all t, as required.

c) $\Rightarrow$ d)

__Lemma:__ $p \in (M, g)$, $\bar{B}_p(r) = \exp(\bar{B}_0(r))$ embedded. [comment on meaning of notation]

If $q \notin \bar{B}_p(r)$, then $\exists\, p' \in \partial \bar{B}_p(r)$ w/

$$d_g(p,q) = d(p,p') + d(p',q)$$



Pf: Let c join p to q. Then

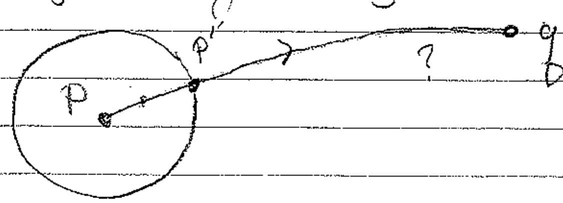
$$L(c) \geq r + d(q',q) \geq r + d(\partial \bar{B}_p(r), q)$$

Thus

$$d(p,q) = \inf_c L(c) \geq r + d(\partial \bar{B}_p(r), q)$$

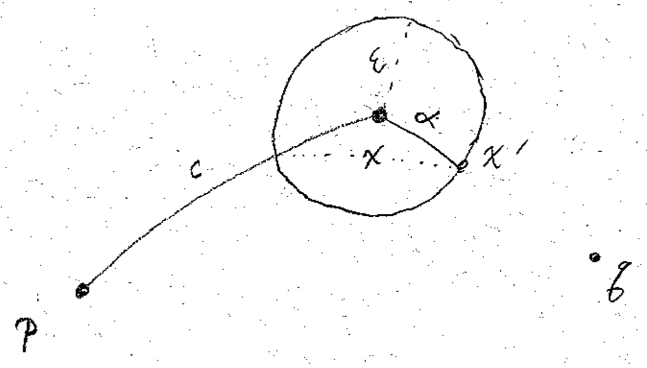
By Δ-$\neq$, have $=$ here. By cptness of $\partial \bar{B}_p(r)$, $\exists\, p'$ sat lemma. ∎

Returning to c) $\Rightarrow$ d). Choose r sat lemma, p' as in conclusion. Let $c : [0, \infty) \to M$ be the geod ray starting at p and passing through p'.



Let $I = \{ t \in [0, d(p,q)] \mid t + d(c(t), q) = d(p,q) \}$.

Note I is clsd and $[0, r] \subseteq I$. <u>Claim:</u> I is open. Let $t \in I$



Set $x = c(t)$. Choose ε s.t. $\overline{B}_x(\varepsilon)$ is embed. By lemma, choose $x' \in \partial \overline{B}_x(\varepsilon)$ s.t.

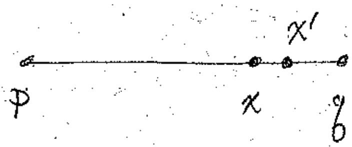
$$d(x,q) = d(x,x') + d(x',q).$$

Let α be geod from x to x'.

Now $d(p,x) = t$ as $d(p,q) \leq d(p,x) + d(x,q) \leq t + d(x,q)$

So

$$d(p,q) = d(p,x) + d(x,q)$$



By triangle ineq, $d(p,x') = d(p,x) + d(x,x') = t + \varepsilon$.

Then α must lie on c, as otherwise, $\exists$ a path of len $< t+\varepsilon$ from p to x'!

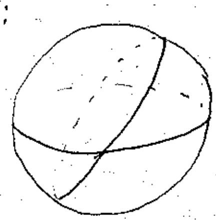
So $[t, t+\varepsilon) \subseteq I$. Thus $I = [0, d(p,q)]$ and we have $c(d(p,q)) = q$. ∎

So $\exists$ a <u>min geod</u> joining p to q.

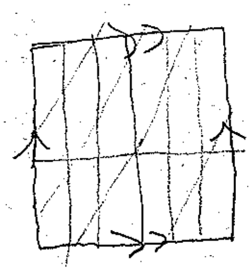
<u>Periodic Geodesics:</u> $\alpha : S^1 \to (M, g)$ local geod

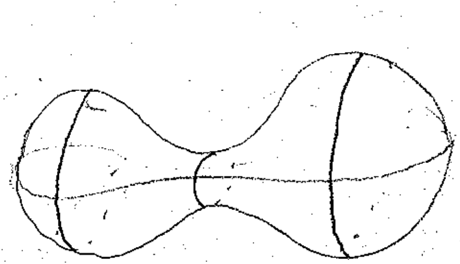
$$\alpha : [a,b] \to M, \ \alpha(a) = \alpha(b), \ \alpha'(a) = \alpha'(b)$$

S^2:

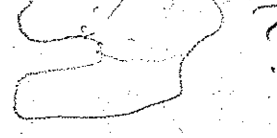


<u>Flat tori:</u>





parallel guys, inf many non parallel fams.



Suppose we have some topology: $\pi_1(M,p) \neq 0$. Lecture 7.

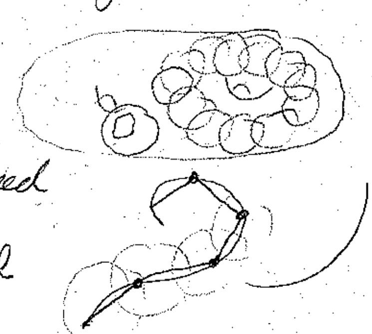
$$\begin{matrix} S^1 \to M \\ \text{up to free} \\ \text{homotopy} \end{matrix} = \begin{matrix} \text{conj classes} \\ \text{in} \\ \pi_1(M,p). \end{matrix}$$

Start on next page, then come back.

<u>Thm</u>: (M,g) a cpt R-mfld. Let $\mathcal{C}$ be a <u>nontrivial</u> homotopy class of loops in M. Then $\exists$ a closed geod in $\mathcal{C}$.

[Idea: curve shortening]

<u>Pf</u>: As M is cpt, $\exists \varepsilon$ s.t. $B_p(\varepsilon)$ is embedded for all p. If $c \in \mathcal{C}$ then $L(c) \geq \varepsilon$, as otherwise c is null homotopic.



Let $l = \inf\{L(c) \mid c \in \mathcal{C}\} \geq \varepsilon$, choose smooth const speed curves $c_i : [0,1] \to M$ in $\mathcal{C}$ w/ $\lim L(c_i) = l$ and $L(c_i) \leq 2l$

Let $0 = t_0 < t_1 < \dots < t_n = 1$ be a partition of $[0,1]$ s.t. $|t_j - t_{j-1}| < \varepsilon/4l$. Note $c_i|_{[t_j, t_{j+1}]}$ has len $< \varepsilon/2$ lies in one of $B(\varepsilon)$

Passing to a subseq, assume $c_i(t_j) \longrightarrow x_j$ for all i.

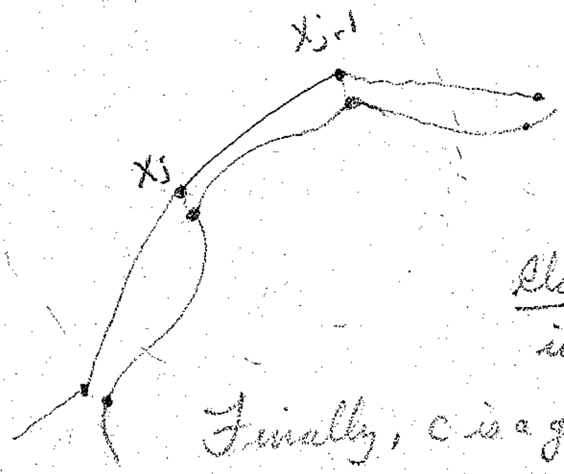
Note $d(x_j, x_{j+1}) < \varepsilon$. Let c be the piecewise geod path joining the x_j

<u>Claim</u>: $L(c) = l$. Choose i s.t. $d(x_j, c_i(t_j)) < \delta$ for all j. Then

$$d(x_j, x_{j+1}) \leq 2\delta + L(c_i|_{[t_j, t_{j+1}]})$$

$$\Rightarrow L(c) = \Sigma \leq 2\delta n + L(c_i)$$

Let $\delta \to 0$, $i \to \infty \Rightarrow L(c) = l$.

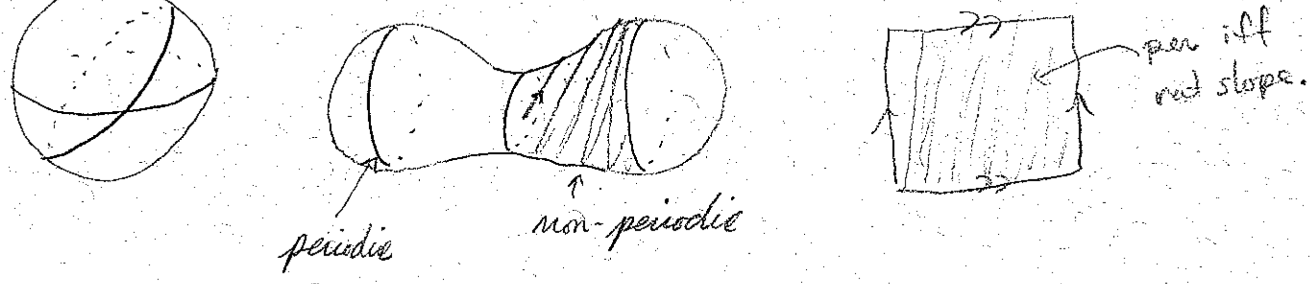


<u>Claim</u>: $c \in \mathcal{C}$. Clear since loc everything lies in $B(\varepsilon)$

Finally, c is a geod as if not can round corners to get $\bar{c} \in \mathcal{C}$ with $L(\bar{c}) < l$. ∎

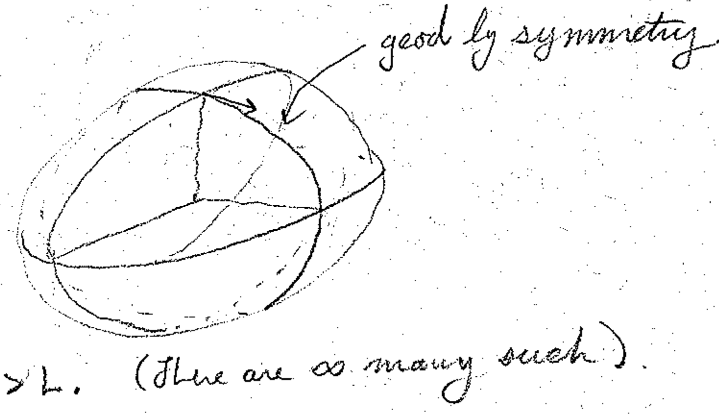
Lecture 7: [Recall:]

Periodic Geodesics: $S^1 \to (M,g)$ locally geod.



may not be embedded (= simple)

Ellipsoid: $x^2/\alpha + y^2/\beta + z^2/\gamma = 1$



Morse (1934): Let $L > 0$. Then $\exists \varepsilon > 0$ s.t. on any ellipsoid w/ axis lengths in $(1, 1+\varepsilon)$ then every period geod other than the 3 shown has length $> L$. (There are ∞ many such).

Zoll surfaces: $\exists$ non-round metrics on S^2 s.t. every geodesic is periodic and simple with length 2π. [These are surfaces of revolution]

[Go back to last page and do theorem]

Thm: (Bangert-Franks-Hingston 1993) Let g be any R-metric on S^2. Then $\exists$ ∞-many distinct periodic geodesics

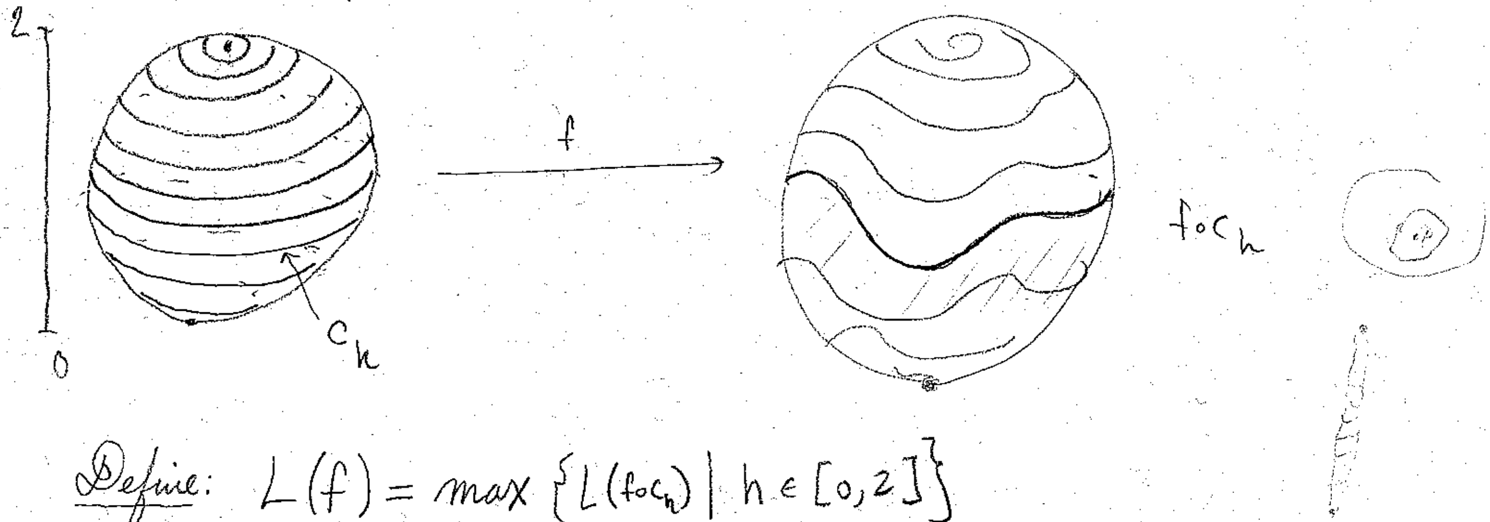
switch

Thm: (Lusternik-Fet) Let M be cpt and $\pi_1 = 1$. Then M has at least one clsd geodesic.

Thm (Birkhoff 1917) Let g be any R metric on S^2. Then $\exists$ a periodic geodesic.

Idea of proof: Minmax arguement.

Let $\mathcal{S} = \{ f : S^2 \to S^2 \mid$ smooth and homotopic to the ident$\}$

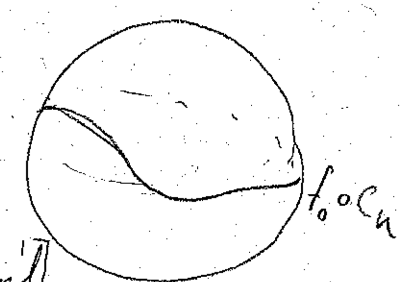


Define: $L(f) = \max\{ L(f \circ c_h) \mid h \in [0,2] \}$

Again, can bound $L(f)$ from below, independent of f.

Suppose $\exists f_0$ s.t. $L(f_0) = \min\limits_{f \in \mathcal{S}} L(f)$.

Then c_h for h w/ $L(f_0 \circ c_h) = L(f_0)$ is a geodesic.



Problem: does f_0 exist? [is limit smooth?, infinite dim'l space.]

Solution: approx by broken geodesics, as before.

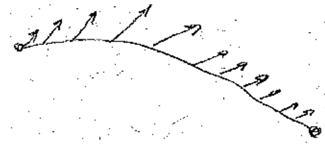
Basic Top input: $\exists$ a non-trivial path of loops in S^2

M cpt mfld. $\Lambda M = \{ c : S^1 \to M \mid \text{conts} \}$ w/ cpt-open top.

[think of as a smooth manifold, will approx by one]

$E : \Lambda M \to \mathbb{R}^+$ via $E(c) = \int_{S^1} \|c'(t)\|^2 dt$

Critical points of E are geodesics.



(critical points have const speed as

$t^2+(1-t)^2$ on $[0,1]$ is min by $t=1/2$)

fibration, like a bundle from homotopy point of view

$$\Omega M \longrightarrow \Lambda M \longrightarrow M$$

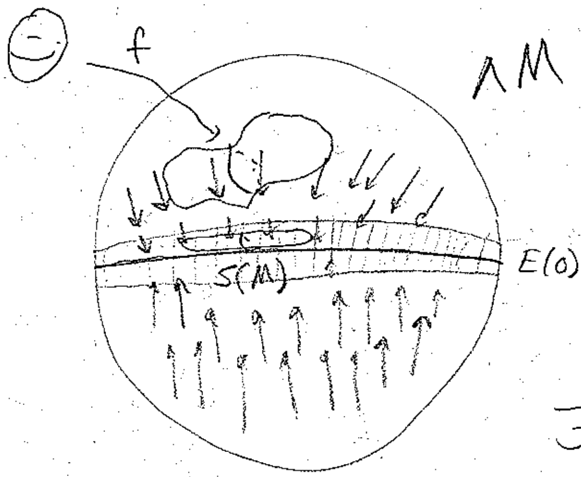
$$\alpha \longmapsto \alpha(0) \quad \text{evaluation}$$

Also have a section $\Lambda M \xleftarrow{s} M$ $\Rightarrow \pi_q(\Lambda M) = \pi_q(M) \oplus \pi_q(\Omega M)$

const loop at $p \longleftarrow p$

$= \pi_q(M) \oplus \pi_{q+1}(M)$

Alg. top shows: if $q>0$ is the first q for which $\pi_q(M) \neq 0$ then $\pi_{q-1}(\Lambda M) \neq 0$.



ΛM

Need E has a critical pt not in $s(M)$. If not, look at $-\mathrm{grad}(E)$.

Consider $f: S^{q-1} \to \Lambda M \neq \pi_{q-1}(\Lambda M)$

Flow by $-\mathrm{grad}(E)$ to homotope f into a nbhd of $s(M)$. Then can push f into $s(M)$. But $\pi_{q-1}(M)=0$, so $f=0$ in $\pi_{q-1}(\Lambda M)$ a contradiction. So E has annother critical point, and so M has a periodic geodesic. ∎

[In real life, approx ΛM by $P_n(\varepsilon)$ consisting of piecewise geodesic polygons w/ $\sum \text{lens}^2 \leq \varepsilon$. For large n $P_n \to \Lambda M$ is an isom on π_q for $q \leq Q$. See R. Bott: Lectures on Morse theory, old and new]

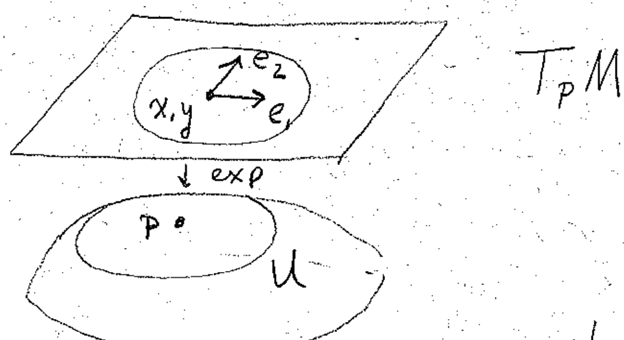
In general, try to get more info about geod from understanding ΛM, ΩM, e.g. computing its homology. } gen prob: showing period geod are distinct.

<u>Serre 1951</u>: M cpt mfld. Then $\exists$ ∞-many k for which $H_k(\Omega M, \mathbb{R}) \neq 0$.

<u>Cor</u>: (M, g) compact, p and q in M. Then $\exists$ ∞-many distinct geodesics joining p to q.

<u>Ex</u>: S^2 

<u>Normal Coordinates.</u>



e_1, e_2 orthonormal basis for T_pM.

Coor on T_pM are $(x, y) \longleftrightarrow xe_1 + ye_2$

<u>Normal coor</u>

On embedded $B_p(\varepsilon)$, these coor push down to coor on a nbhd of p in M.

x, y coor on U, $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$

[Remarkable prop — every R-mfld is locally Euclidean to <u>2nd</u> order]

On U:

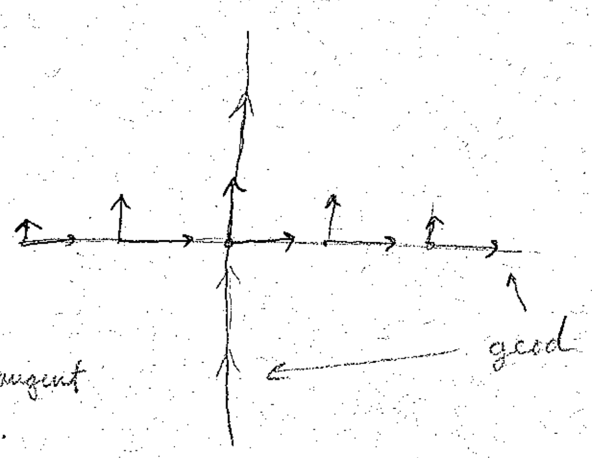
$$g = a(x,y)\, dx^2 + c(x,y)\,(dx\,dy + dy\,dx) + b(x,y)\, dy^2$$

Taylor series about $(0,0) = p$

$$a(x,y) = a(\vec{0}) + \frac{\partial a}{\partial x}(\vec{0})\, x + \frac{\partial a}{\partial y}(\vec{0})\, y + \frac{1}{2}\frac{\partial^2 a}{\partial x^2} x^2 + \frac{\partial^2 a}{\partial x \partial y} xy + \frac{1}{2}\frac{\partial^2 a}{\partial y^2} y^2 + o(x^2 + y^2)$$

[$o(x^2+y^2)$ means]

Note $a = g(X,X)$



So

$a(\vec{0}) = 1$, $\frac{\partial a}{\partial x}(\vec{0}) = 0$ as X is unit tangent along the geod.

$\frac{\partial a}{\partial y}(\vec{o}) = Y g(X,X)\Big|_p = 2g(D_Y X, X)\Big|_p = 0$ by

<u>Key Pt</u>: In normal coor $D_Y X$ and $D_X Y$ vanish at p:

$$\underset{\text{as all rays are geod.}}{0} = D_{X+Y} X+Y = \underset{0}{\cancel{D_X X}} + \underbrace{D_X Y + D_Y X}_{\text{are equal by symmetry.}} + \underset{0}{\cancel{D_Y Y}} \Rightarrow D_X Y\big|_p = D_Y X\big|_p = 0$$

$\frac{\partial^2 a}{\partial x}(\vec{o}) = 0 \qquad a|_{x\text{-axis}}$ const

$\frac{\partial^2 a}{\partial y \partial x}(\vec{o}) = Y(X(g(X,X))) = Y 2g(D_X X, X) = 2g(D_Y D_X X, X) = 2g(D_Y D_X X, X)$

$\frac{\partial^2 a}{\partial y^2} = Y Y g(X,X) = 2g(D_Y D_Y X, X)$

Thus

$a = 1 + g(D_Y D_Y X, X)(2xy + y^2) \qquad + o(x^2+y^2)$

$b = 1 + g(D_X D_X Y, Y)(2xy + x^2) \qquad + o(x^2+y^2)$

$c = \frac{1}{2} g(D_X D_X Y, X)x^2 + \big(g(D_Y D_X X, Y) + g(D_Y D_Y X, X)\big)xy + \frac{1}{2} g(D_Y D_Y X, Y)y^2$
$+ o(x^2+y^2)$

Lecture 8: $\boxed{\text{Curvature 101}}$

On a surface: chose normal coordinates (x,y) about $p=(0,0)$

$$g = (1+O(r^2))\,dx^2 + (0+O(r^2))\,(dx\,dy+dy\,dx) + (1+O(r^2))\,dy^2$$

where $r=\sqrt{x^2+y^2}$ and $f(x,y)$ is $O(r^2)$ if $\exists$ a const C

s.t. $|f(x,y)|< Cr^2$ for all (x,y) close to $(0,0)$.

Thus if $V\in T_{(x,y)}M$ is $v_1\frac{\partial}{\partial x}+v_2\frac{\partial}{\partial y}$ we have

these are fns of (x,y)

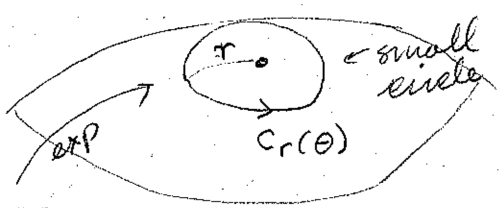
$$\|V\|_g^2 = \|V\|^2_{Euclid} + \left(O(r^2)\,v_1^2 + O(r^2)\,2v_1v_2 + O(r^2)\,v_2^2\right)$$

$$= \|V\|^2_{Euclid}\left(1+O(r^2)\right) \qquad \left[=\|V\|^2_{Euclid}\big(1+\underbrace{E(x,y,v_1,v_2)}_{|\ |<Cr^2}\big)\right]$$

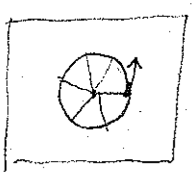
Thus

$$\|V\|_g = \|V\|_{Euclid}\left(1+O(r^2)\right) \qquad \left[\sqrt{\ }\text{ is being eval near non-sing point } 1\right]$$

Consider



$$\frac{L(c_r)}{2\pi r} = \frac{1}{2\pi}\int_0^{2\pi}\frac{\|c_r'(\theta)\|_g}{r}\,d\theta \qquad r = \|c_r'(\theta)\|_{Euclid}$$



$$= \frac{1}{2\pi}\int_0^{2\pi}(1+O(r^2))\,d\theta = 1+O(r^2)$$

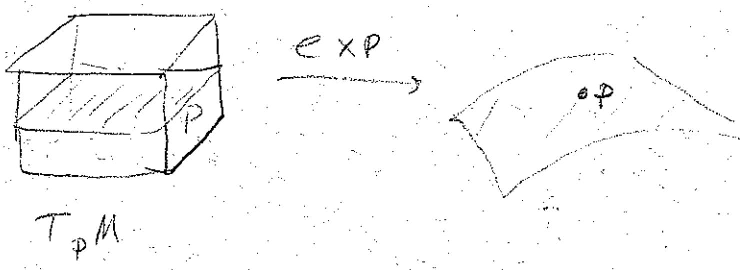
So $L(c_r)=2\pi r(1+O(r^2))$. Assuming there is a 2^{nd} order term define K by $L(c_r)=2\pi r\left(1-\frac{K}{6}r^2+O(r^3)\right)$

or $K=\lim_{r\to 0} -\frac{1}{6}\left(\frac{L(c_r)}{2\pi r}-1\right)$ $\qquad$ [issue: don't know $L(c_r)$ is a smooth fn of r]

(Gauss Curvature)

<u>Sectional Curvature</u>: P plane in T_pM ($P \in G_p^2 M$ - 2 grassmanian)

$K(P)$ = gauss curvature of $\exp(P)$ at p.



[This is how curv was introduced by Riemann in orig paper. But how to compute, right context etc]

<u>Curvature Tensor</u>: on (M, g) is the $(1,3)$ tensor defined by

$$R_p(x,y,z) = R_p(x,y)z = D_Y(D_X Z) - D_X(D_Y Z) + D_{[X,Y]} Z.$$

<u>where</u> X, Y, Z are vector fields w/ vals x, y, z at p

- Check is C^∞_1, linear in all 3-variables.
- chn local coordinates, $[X, Y] = 0$, measuring failure of parallel transport to commute.

- For $\mathbb{R}^n$, $R_p = 0$ as $R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = 0$ [will see this char locally Euclidean metrics]
- notice $D_Y D_X Z$ terms similar to normal coor last time.

<u>Associated $(0,4)$ tensor</u>:

$$R(a,b,c,d) = g(R(a,b)c, d) \in \mathbb{R}$$

<u>Alg props</u>: $\forall x, y, z, w \in T_pM$ have

i) $R(a,b,c,d) = -R(b,a,c,d) = -R(a,b,d,c)$

ii) $R(a,b)c + R(c,a)b + R(b,c)a = 0$
$R(a,b,c,d) + R(c,a,b,d) + R(b,c,a,d) = 0$ } Bianchi identity

iii) $R(a,b,c,d) = R(c,d,a,b)$

Pf: i) first = clear from formula for $R(a,b)c$.

second follows if $R(a,b,c,c)=0$. Choose A,B,C w $[A,B]=0$ and $g(C,C)=1$

$$R(a,b,c,c) = g(D_B D_A C - D_A D_B C, C)$$

Now

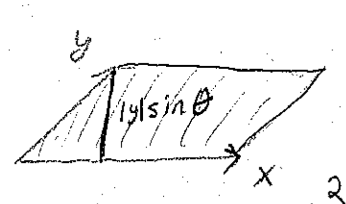
$$g(D_B D_A C, C) = B\, g(D_A C, C) - g(D_A C, D_B C)$$
$$= \tfrac{1}{2} B(A\, \cancel{g(C,C)}) - g(D_A C, D_B C) = -g(D_A C, D_B C)$$

Other term is same via sym, so $R(a,b,c,c)=0$.

ii) straight forward: use pairwise commuting variables, or Jacobi identity. ▨

iii) HW.

<u>Fancy Language</u>: i) and iii) mean that R is a symmetric bilinear form on $\Lambda^2 T_p M = \{\sum v_i \wedge w_i\}$



For $x, y \in T_p M$ define

$|x|^2|y|^2\cos\theta$

$$|x\wedge y|^2 = g(x,x)g(y,y) - g(x,y)^2 = \begin{pmatrix}\text{area of parallelogram}\\ \text{spanned by } x, y\end{pmatrix}^2$$

[could discuss curv. op. at this point]

For a plane P in $G^2 T_p M$, set

$$K(P) = \frac{R(x,y,x,y)}{|x\wedge y|^2} \quad \text{where is any basis basis for } P.$$

Called the sectional curvature; owe a proof that it is same as at beginning.

To check well defined check under basic ops: $\{X,Y\} \to \{Y, X\}$, $\{X,Y\} \to \{\lambda X, Y\}$, $\{X,Y\} \to \{X+\lambda Y, Y\}$

Alt, use that R is defined on $\Lambda^2 T_pM$ and that $\Lambda^2 P$ is one dimensional and formula behaves correctly under scaling.

<u>Thm</u> (GHL): The sectional curvatures determine R.

<u>Ex</u>: dim 2

$$R(a,b,c,d) = K(T_pM) \begin{vmatrix} g(a,c) & g(a,d) \\ g(b,c) & g(b,d) \end{vmatrix}$$

scalar prod on Λ^2 assoc to g.

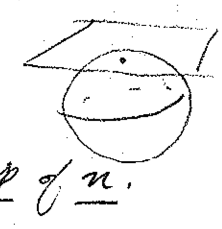
[can also deduce from the fact that Λ^2 is 1-dim in this case]

<u>Ex</u>: $\mathbb{R}^n$, $R = 0$, $K(P) = 0$. Same for flat tori.

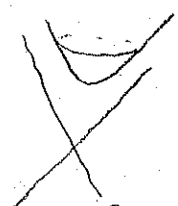
<u>Ex</u>: S^2: homogeneous as $O(3)$ acts trans by isom $\Rightarrow K(P)$ is constant

S^n: hom under $O(n+1)$, moreover $\mathrm{Stab}(p)$ is $O(n)$

so action is trans on G^2TS^n. $\Rightarrow K(P)$ is const, <u>indep of n</u>.



Will see const is $+1$



$-X_0^2 + X_1^2 + X_2^2 + \cdots + X_n^2$

<u>Ex</u>: $\mathbb{H}^n$: hyperbolic space

$SO_0(n+1)$ acts trans on $G^2T\mathbb{H}^n \Rightarrow K(P)$ is const (turns out -1)

[will see these are the unique complete simply connected guys of const curvature,]

<u>Ex</u>: $\mathbb{R}P^n$: const curv, same as S^2; lens spaces.

Prop: G Lie gp with a biinvariant metric. Then if A, B, C, D are left inv vector fields. Then $R(A,B,C,D) = \frac{1}{4} g([A,B],[C,D])$

Cor: $K(P) \geq 0$ for all P. $K(P) > 0$ if $\nexists$ non-colinear $A, B \in \mathfrak{g}$ w/ $[A,B]=0$

Pf: By 2.90 HW $D_X Y = \frac{1}{2}[X,Y]$ for left inv X, Y.

$$R(A,B)C = \underbrace{\tfrac{1}{4}[B,[A,C]]}_{= D_B D_A C} - \underbrace{\tfrac{1}{4}[A,[B,C]]}_{= +\frac{1}{4}[A,[C,B]]} + \underbrace{\tfrac{1}{2}[[A,B],C]}_{= 1/2\,[C,[B,A]]}$$

via Jacobi $$= \frac{1}{4}[C,[B,A]] = \frac{1}{4}[[A,B],C];$$ Consider left v.f. X, Y, Z, let $\phi_t = \exp_e(tZ_e)$ the flow assoc to Z. Then by biinvariance,

$$g(Ad_{\phi_t} X_e, Ad_{\phi_t} Y_e) = g(X_e, Y_e).$$

Diff gives
$$g([Z,X],Y) + g(X,[Z,Y]) = 0 \Rightarrow g([X,Z],Y) = g(X,[Z,Y])$$

so

$$R(A,B,C,D) = \frac{1}{4} g([[A,B],C],D) = -\frac{1}{4} g([A,B],[C,D]). \quad \blacksquare$$

Ex: $G = SU(2) = \{ A \in M_2(\mathbb{C}) \mid A\bar{A}^t = I, \det(A) = 1 \}$

$$= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| (a,b) \in \mathbb{C}^2, a\bar{a} + b\bar{b} = 1 \right\} \cong S^3 \subseteq \mathbb{R}^4$$

$$\mathfrak{g} = \left\{ \begin{pmatrix} is & v \\ -\bar{v} & -is \end{pmatrix} \middle| s \in \mathbb{R}, v \in \mathbb{C} \right\}$$ (actually skip, this and follow next page on quaternions

$\mathcal{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$ $ij = -ji = k$
$i^2 = j^2 = k^2 = -\underline{1}$

$\bar{v} = a - bi - cj - dk$; $\overline{vw} = \bar{w}\,\bar{v}$, etc. Hamilton's quaternions.

$|v|^2 = v\bar{v} = a^2 + b^2 + c^2 + d^2 = \bar{v}v$

Set $G = \{v \in \mathcal{H} \mid v\bar{v} = 1\} \cong S^3$ a subgroup by $\overline{vw} = \bar{w}\,\bar{v}$

The metric $|v|^2$ on $\mathcal{H}$ is inv under G on both left and right because of $\overline{vw} = \bar{w}\bar{v}$ (action on $T_w\mathcal{H} \to T_{vw}\mathcal{H}$ is left mult under nat'l ident of both w/ $\mathcal{H}$). Thus the round metric on G is biinvariant. Now $e \in G$ is 1

so $T_eG = \text{span}(i, j, k)$. Set I, J left inv ext of i, j at 1, i.e. $I_v = vi$, $J_v = vj$. Then $[I, J] = 2K$ (Think flow picture, use $= ij - ji = 2k$)

Thus const curve of $S^3 = 1$.

Another name for G is $SU(2) = \left\{A \in M_2(\mathbb{C}) \;\middle|\; \begin{matrix} A\bar{A}^t = I \\ \det(A) = 1 \end{matrix}\right\}$

Lecture 9: Last time $R(a,b,c,d)$, $K(P)$, etc.

Simplifications:

$$\text{Average sect curv at } p = \int_{G^2 T_pM} K(P)\, dV$$

$$= \frac{1}{n(n-1)} \underbrace{\sum_{i,j=1}^{n} K(e_i, e_j)}_{\mathrm{Scal}_p}$$

finite average

where $e_1, \ldots, e_n$ is an orthonormal basis for T_pM

(Pf: $R(v_1, v_2) = \sum_{i,j} a_i^2 b_j^2 R(e_i, e_j)$ if $v_1 = \sum a_i e_i$, $v_2 = \sum b_i e_i$; taking expectations any using symmetry, get $\int K(P) = E(a_1^2 b_2^2)\, \mathrm{Scal}_p$. Comp const by const curv. case.)

$G^2 T_pM$ has a unit mass volume form coming from a metric inv. under the action of $O(T_pM, g) \cong O(n)$.

$G^2 T_pM \cong O(n)/O(2) \times O(n-2)$

Up to scale, any left inv R-metric on $O(n)$ induces the same volume form.

Note: for us $\mathrm{Scal}(S^n) = n(n-1)$; some authors norm so $\mathrm{Scal}(S^n) = 1$

Ricci Curvature: $x, y \in T_pM$, define $F: T_pM \circlearrowleft$ by $v \mapsto R(x,v)y$

End of v. sp, so can take $\mathrm{trace}(F) = \mathrm{Ric}_p(x,y)$. If e_i are a orthonormal basis, we have

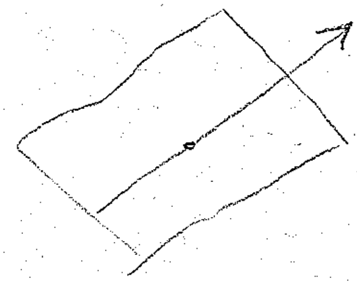
$$v \xmapsto[F]{} \sum_i R(x, v, y, e_i)\, e_i$$

So taking the trace get

$$\mathrm{Ric}_p(x,y) = \sum_{i=1}^{n} R(x, e_i, y, e_i)$$ — a symmetric bilinear form

Has a geom meaning

$$\mathrm{Ric}_p(x,x) = \begin{matrix}\text{Average over} \\ P \in G^2 T_pM \\ \text{containing } x\end{matrix} \text{ of } K(P)$$



Also:
$X \longmapsto Ric_p(X, \cdot) \in T_p^*M \longrightarrow T_pM$ gives an endo. of T_pM.

Trace is $Scal_p$.

Important Class: Einstein Metrics $g = \lambda Ric$ at each p.

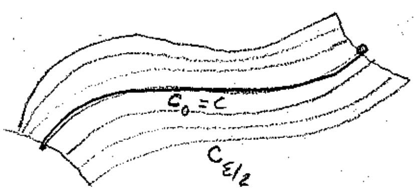
Ex: const curv (clear from geometric picture)

For Lorentzian metrics, Einstein's eqn in a vacuum is $Ric = 0$.

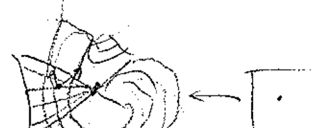
[Mention Poincare Conj, Perelman, if time permits]

Variation of Arc Length and connections to curvature:

Def: $c : [a,b] \to M$ smooth curve. A variation of c is a smooth map $H : [a,b] \times (-\varepsilon, \varepsilon) \to M$ s.t. (variables s, t)
$H(s,0) = c(s)$. Let $c_t : [a,b]$ by $c_t(s) = H(s,t)$



Issue: H may be far from an embedd.



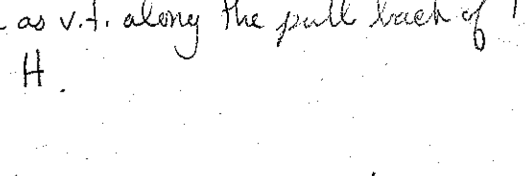
Def: $H : N \to M$ a smooth map. A vector field along H is a smooth map $Y : N \to TM$ s.t. $Y(p) \in T_{H(p)}M$. [same as v.f. along the pull back of TM by H.]
Denoted $\Gamma(H, TM)$.

Thm (M,g) and $H : N \to M$ smooth.
$X \in \Gamma(TN)$, $Y \in \Gamma(H, TM)$, $\exists$ a connection $\bar{D}_X Y$ on such pairs sat.

i) bilinear, ii) $\bar{D}_{fX} Y = f \bar{D}_X Y$, $\bar{D}_X(fY) = X(f)Y + f\bar{D}_X Y$ $f \in C^\infty(N)$

iii) locally compat w/ metric conn D on M.

$$(\bar{D}_X Y)_p = \left(D_{H_*(X_p)} \tilde{Y}\right)_{H(p)}$$



Pf: compute in local coor, or use diff def of connection.

$X \in \Gamma(TN) \longrightarrow \overline{X} \in \Gamma(H, TM)$ via push forward of H.

Prop: $X, Y \in \Gamma(TN)$, $U, V \in \Gamma(H, TM)$. Then

i) $\overline{D}_X \overline{Y} - \overline{D}_Y \overline{X} = \overline{[X,Y]}$

ii) $X\, g(U,V) = g(\overline{D}_X U, V) + g(U, \overline{D}_X V)$

iii) $\overline{D}_Y(\overline{D}_X U) - \overline{D}_X(\overline{D}_Y U) + \overline{D}_{[X,Y]} U = R(\overline{X}, \overline{Y})U$ R - curv. tensor

Pf: Show diffs are tensors, then check for coor v.f. on N and M.

Cor: The Gauß lemma holds true in general,  $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$ remain $\perp$ under exp. proof is formally the same. (exp not always an immersion: think S^2).

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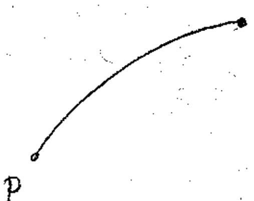
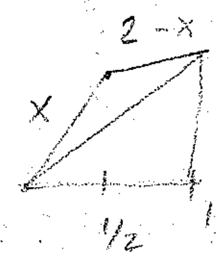
c a curve. Set

$$E(c) = \int_a^b \|c'(s)\|^2 ds$$

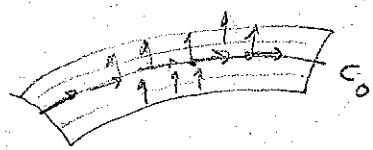


Suppose $c : [0,1] \longrightarrow M$ joins p to q w/ minimal energy. Then c is minimal length (use const speed ☆ param).

Pf: Consider min energy over reparam. of a fixed path. As $x^2 + (2-x)^2$ is min by $x = 1$ lowest energy reparam is constant speed. In which case, $E(c) = L^2$ (or $L^2/(b-a)$). Thus min $E \Rightarrow$ min L.



H a var of a curve $c : [a,b] \to M$



get v.f. $T = H_*\left(\frac{\partial}{\partial t}\right)$ along c —

$= \overline{\frac{\partial}{\partial t}}$ get "infinitessimal var" $T(s)$

[Can go other way, start w/ v.f. along c, get variation; e.g. use exp.]

<u>First Var Formula</u>: H var of c, have

$$\frac{1}{2}\frac{d}{dt}E(c_t)\bigg|_{t=0} = g(T(s), c'(s))\bigg|_a^b - \int_a^b g(T(s), D_{c'}c'(s))\,ds$$

<u>Cor</u>: Let Ω_{pq} = curves $c : [0,1] \to M$ joining p to q.
Then the <u>critical points</u> of $E : \Omega_{pq} \to \mathbb{R}$ are exactly the geodesics
means

<u>Pf</u>: $c \in \Omega_{pq}$ is a crit pt iff $\forall$ var H we have $\frac{d}{dt}E(c_t)\Big|_{t=0} = 0$

By form, $\Longleftrightarrow \int_0^1 g(T(s), D_{c'}c'(s))\,ds = 0$ for all vector fields $T(s)$ along c ∎

$\Longleftrightarrow D_{c'}c'(s) = 0$ for all s.

<u>Proof of formula</u>: Let $T = H_*\left(\frac{\partial}{\partial t}\right)$, $S = H_*\left(\frac{\partial}{\partial s}\right)$

$$\frac{d}{dt}E(c_t) = \frac{d}{dt}\int_a^b g(c_t'(s), c_t'(s))\,ds = \int_a^b \frac{d}{dt}g(c_t'(s), c_t'(s))\,ds$$

$$= \int_a^b \frac{\partial}{\partial t}g(S,S)\,ds = \int_a^b 2\,g(\overline{D}_{\frac{\partial}{\partial t}}S, S)\,ds$$

(by symmetry) $= 2\int_a^b g(\overline{D}_{\frac{\partial}{\partial s}}T, S)\,ds = 2\int_a^b \frac{\partial}{\partial s}g(T,S) - g(T, \overline{D}_{\frac{\partial}{\partial s}}S)\,ds$

$$= 2\,g(T, c')\Big|_a^b - 2\int_a^b g(T(s), D_{c'}c'(s))\,ds.$$ ∎

[useful for deciding when curve is a geodesic]

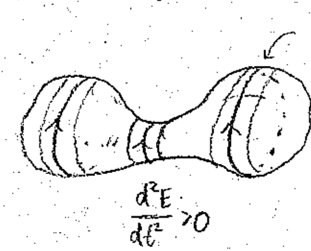
Also have formula for var of length, when c_0 has unit speed param

$$\frac{d}{dt} E(c_t)\Big|_{t=0} = 2 \frac{d}{dt} L(c_t)\Big|_{t=0}$$

$$\left[\frac{d}{dt} \int g(S,S)^{1/2}\, ds = \frac{1}{2}\int \overset{1}{g(S,S)}^{-1/2} g(D_S T, S) = \frac{1}{2}\int g(S,S)^{-1/2}\left(S g(T,V) - g(T, D_S \tfrac{d}{ds}\right.\right.$$

$\boxed{\text{Lecture 10}}$

[If a geodesic, what is second order term?]



$c : [a,b] \to M$ a length parametrized geodesic.

Then set $T = H_*(\frac{\partial}{\partial t})$, $S = H_*(\frac{\partial}{\partial s})$. Will abuse notation, $S = \frac{\partial}{\partial s}$, no $\bar{D}$ etc.

2nd Variation Formula: $$\frac{1}{2}\frac{d^2}{dt^2} E(c_t)\Big|_{t=0} = g(D_T T, S)\Big|_a^b + \int_{c_0} |D_S T|^2 - R(T,S,T,S)\, ds$$

$$D_T D_S T = R(S,T)T + D_S D_T T$$

Pf: Again, diff under the integral sign

$$\frac{d^2}{dt^2} g(S,S) = \frac{d}{dt}\, 2g(D_T S, S) = 2g(D_T D_T S, S) + 2g(D_T S, D_T S)$$

$$\frac{1}{2}\frac{d^2}{dt^2} g(S,S) = \underset{\overset{\shortparallel}{S\, g(D_T T, S)}}{g(D_S D_T T, S)} - R(S,T,S,T) + |D_T S|^2$$

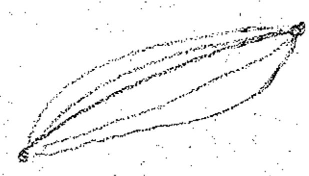
$S\, g(D_T T, S)$ for $t=0$ as is a geod.

integrate to get result.

Alt form: Suppose T vanishes at end pt of c. Then

$$\frac{1}{2}\frac{d^2}{dt^2} E(c_t)\Big|_{t=0} = \int_{c_0} |D_S T|^2 - R(T,S,T,S)\, ds = -\int g(T, D_S D_S T + R(S,T)S)\, ds$$

Pf: $g(D_S T, D_S T) = \underbrace{S\, g(T, D_S T)}_{\int \text{ vanishes because } \partial \text{ terms } 0} - g(T, D_S D_S T)$.



<u>Synge Thm</u>: M^n cpt orient <u>even dim</u> R-mfld.

If $K(P) > 0$ everywhere, then $\pi_1 M = 1$.

Ex: S^{2n}, $\mathbb{C}P^n$

Note: • even dim need because $\mathbb{R}P^n$ is orient in odd dims (still π_1 is finite)

• orient needed because of $\mathbb{R}P^n$ ($\pi_1 = \mathbb{Z}/2$, non orient in even dims)

• $K(P) > 0$ needed because of flat-n-tori, $S^{n-1} \times S^1$

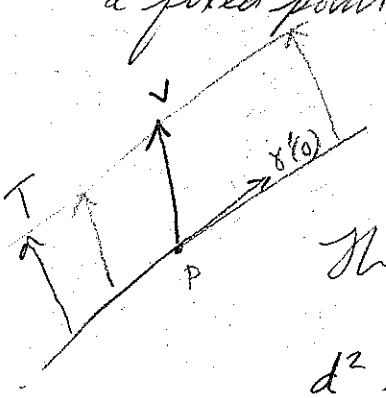
[see intuit at bottom]

Pf: Suppose $\alpha \in \pi_1 M$ is not 1. Choose γ which has min. energy among all maps $\gamma : [0,1] \to M$ w/ $\gamma(0) = \gamma(1)$ rep α. [showed exist of such before]

Know γ is a geod. Let $p = \gamma(0)$, let $P : T_pM \circlearrowleft$ be parallel transport along γ. Let $S_\perp = \{ v \in T_pM \mid \|v\| = 1 \text{ and } g(v, \gamma'(0)) = 0 \} \cong S^{n-2}$

As P is an isometry sending $\gamma'(0) \to \gamma'(0)$, $P(S_\perp) = S_\perp$ and <u>preserves the orient of $S_\perp^{n-2}$.</u>

Alg top says that as $n-2$ is even, $P : S_\perp^{n-2} \circlearrowleft$ must have a fixed point, say v. (or look at evalues) Let T be the vector field along γ which is the parallel tran of v.



Let H be a var of γ w/ inf. def T so that each c_t is a loop.

Then by 2^{nd} var formula, we have

$$\frac{d^2}{dt^2} E(c_t) \Big|_{t=0} = - \int_{c_0 = \gamma} R(T, S, T, S) < 0$$

Thus $\exists\, c_t$ w/ $E(c_t) < E(\gamma)$. But c_t is homotopic to γ, contradicting the minimality of γ. ∎

[at some pt, give intuition: non-twisting T leads for S^n like pic, w/ geodesics coming together.]

Myers' Thm: (M, g) a complete R-mfld s.t. there is a const $C > 0$ for which $K(P) \geq C$ for all $P \in G^2TM$. Then $\mathrm{diam}(M, d_g) \leq \pi C^{-1/2}$. In particular M is cpt. (via exp. map)

Cor: M cpt R-mfld w/ $K(P) > 0$ everywhere. Then $\pi_1 M$ is finite.

Pf: As M cpt, $\inf K(P)$ is realized $\Rightarrow \exists\, C > 0$ s.t. $K(P) \geq C$ everywhere.
Let $\tilde{M}$ be the univ cover of M; $\tilde{M}$ is complete [prompt class for why] and $K(P) \geq C$ everywhere. So $\tilde{M}$ is cpt $\Rightarrow \tilde{M} \to M$ is finite $\Rightarrow \pi_1 M$ is finite. $\blacksquare$

Why particular constants? $S^n(r)$ = sphere in r in $\mathbb{R}^{n+1}$. [What is curv? explain.]

$S^n(1) \longrightarrow S^n(r)$ by dialation. $d_{g_r} = r\, d_{g_1}$
$\quad g_1 \qquad\quad g_r$

$\mathrm{diam}(S^n(r)) = r\, \mathrm{diam}(S^n(1)) = \underline{\pi r}$

So: $g_r = r^2 g_1$, thus $D_r = D_1$ by HW. $\Rightarrow R(a,b)c$ pos.

So $R_r(a,b,c,d) = r^2$ and $K_r(P) = r^{-2} K_1(P) = r^{-2}$

$\left(K(P) = \frac{R}{(\text{area of } \square)^2} \right)$. So $S^n(r)$ realizes bound in Meyer's thm.

[Discuss comparison geometry point of view:
"if you are at least as curved as $S^2(r)$ then your diam is no larger."]