HW 1 SOLUTIONS, MA525

1. Problem 1

Let C be a closed subset of Y. Then $f^{-1}(C) = f_A^{-1}(C) \cup f_B^{-1}(C)$. Since f_A and f_B are continuous, the sets $f_A^{-1}(C)$ and $f_B^{-1}(C)$ are closed in A and B respectively, and hence in X. This implies $f^{-1}(C)$ is closed in X, and hence f is continuous.

2. Problem 4

$(a) \Rightarrow (b)$. Let $f : S^1 \to X$ be a map and suppose f is homotopic to the constant map $c_0 : S^1 \to x_0$. This means that there is a map $F : S_1 \times [0,1] \to X$ such that $F(a,0) = f(a)$ and $F(a,1) = x_0$. Consider the quotient map $q : S^1 \times [0,1] \to \mathbb{D}^2$ given by identifying all points $(a,1)$ to a single point. Because $F(a,1) = x_0$, the map descends to a map $\overline{F} : \mathbb{D}^2 \to X$ such that $F = \overline{F} \circ q$. The map $\overline{F}$ restricted to $S^1 = q(S^1 \times \{0\})$ is f.

$(b) \Rightarrow (c)$. Embed $S^1 \in \mathbb{C}$ in the standard way. Think of a loop based at x_0 as a map $f : S^1 \to X$ that sends $1 \in S^1$ to x_0. Extend it to a map $F : \mathbb{D}^2 \to X$. Consider the map $G : S^1 \times [0,1] \to X$ defined by $G = F \circ q$. The map G defines a homotopy to the constant map $F(0)$. Since any two constant maps are homotopic and homotopy is a transitive relation, $\pi_1(X, x_0) = 0$.

$(c) \Rightarrow (a)$. Directly from definitions.

When X is simply connected, every map f from S^1 into X thought of as a loop in X based at $x_0 = f(1)$ is homotopic to the constant map to x_0. Similarly, g from S^1 into X is homotopic to the constant map to $g(1)$. Any two constant maps are homotopic. Since homotopy is a transitive relation, this implies f and g are homotopic. The converse is just the definition.