HW 10 SOLUTIONS, MA525

Hatcher 2.2 Problem 29

The Mayer-Vietoris sequence gives

$$\cdots \to H_n(M_g) \to H_n(R) \oplus H_n(R) \to H_n(X) \to H_{n-1}(M_g) \to \cdots$$

The compact space R deformation retracts on a wedge of g circles. So $H_0(R) = \mathbb{Z}$, $H_1(R) = \oplus_{i=1}^{g}\mathbb{Z}$, $H_n(R) = 0$ for $n > 1$. Also we know $H_0(M_g) = \mathbb{Z}$, $H_1(M_g) = \oplus_{i=1}^{2g}\mathbb{Z}$, $H_2(M_g) = \mathbb{Z}$, $H_n(R) = 0$ for $n > 1$. Call the 1-cycles on M_g that give the generators of $H_1(R)$ as the a-cycles. Then, there are the 1-cycles on M_g that bound disks in R, which we shall call the b-cycles. The a

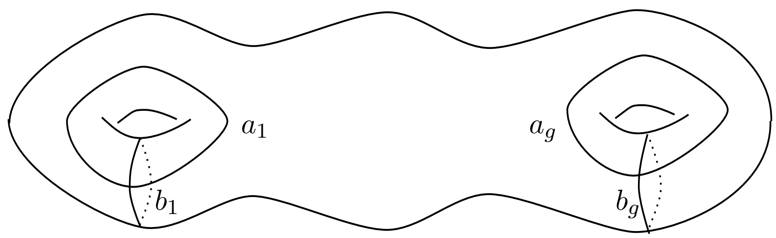

Figure 1. M_g

and b-cycles together generate $H_1(M_g)$ and the induced maps $H_1(M_g) \to H_1(R)$ send the a-cycles to the generators of $H_1(R)$ for both the handle-bodies and send the b-cycles to zero. By computing the various maps in the Mayer-Vietoris sequence, in particular, the map $H_1(M_g) \to H_1(R) \oplus H_1(R)$ has the diagonal as the image, with kernel $\oplus_{i=1}^{g}\mathbb{Z}$ generated by the b-cycles, we get

(1) Since $H_n(R) = H_n(M_g) = 0$ for all $n \geq 3$, we get $H_n(X) = 0$ for all $n > 3$.
(2) Because $H_3(R) = H_2(R) = 0$, we get $H_3(X) = H_2(M_g) = \mathbb{Z}$.
(3) Because $H_2(R) = 0$, we have the exact sequence

$$0 \to H_2(X) \to H_1(M_g) \to H_1(R) \oplus H_1(R) \to \cdots$$

which implies $H_2(X) = Ker(H_1(M_g) \to H_1(R) \oplus H_1(R)) = \oplus_{i=1}^{g}\mathbb{Z}$ generated by the b-cycles.
(4) The Mayer-Vietoris sequence terminates in

$$\cdots \to H_1(X) \to H_0(M_g) \to H_0(R) \oplus H_0(R) \to H_0(X) \to 0$$

This implies the map $H_0(R) \oplus H_0(R) \to H_0(X)$ is surjective, and as before the map $H_0(M_g) \to H_0(R) \oplus H_0(R)$ has image the diagonal. Hence $H_0(X) = H_0(R) \oplus H_0(R)/\Delta = \mathbb{Z}$.
(5) The map $H_0(M_g) \to H_0(R) \oplus H_0(R)$ is also injective, so the image of $H_1(X) \to H_0(M_g)$ is just 0. This gives us the exact sequence

$$0 \to H_2(X) \to H_1(M_g) \to H_1(R) \oplus H_1(R) \to H_1(X) \to 0$$

In the above sequence, the image of $H_1(M_g)$ is the diagonal. So $H_1(X) = H_1(R) \oplus H_1(R)/\Delta = \oplus_{i=1}^{g}\mathbb{Z}$.

To find the relative homology groups $H_n(R, M_g)$, we use the long exact sequence in homology

$$\cdots \to H_n(M_g) \to H_n(R) \to H_n(R, M_g) \to H_{n-1}(M_g) \to \cdots$$

This gives

(1) $H_n(R, M_g) = 0$ for $n > 3$ because $H_n(R) = H_n(M_g) = 0$ for $n \geq 3$.
(2) $H_2(R) = 0$ implies $H_3(R, M_g) = H_2(M_g) = \mathbb{Z}$ and we get the exact sequence

$$0 \to H_2(R, M_g) \to H_1(M_g) \to H_1(R) \to \cdots$$

which means that $H_2(R, M_g) \to H_1(M_g)$ is injective.
(3) The long exact sequence terminates in

$$\cdots \to H_1(R, M_g) \to H_0(M_g) \to H_0(R) \to H_0(R, M_g) \to 0$$

The map $H_0(M_g) \to H_0(R)$ is an isomorphism, so $H_0(R, M_g) = 0$. Also, the isomorphism gives the exact sequence

$$0 \to H_2(R, M_g) \to H_1(M_g) \to H_1(R) \to H_1(R, M_g) \to 0$$

The map $H_1(M_g) \to H_1(R)$ is onto, so $H_1(R, M_g) = 0$. So $H_2(R, M_g) = Ker(H_1(M_g) \to H_1(R)) = \oplus_{i=1}^{g} \mathbb{Z}$.

Remark 0.1. Instead of gluing by the two handle-bodies by the identity map of the boundary surface, if we glue by a map of the surface that interchanges the a and b-cycles, we get S^3 (embed the genus g handle-body in $\mathbb{R}^3$ in the standard way; check that the complement of it in S^3 is a genus g handle-body with the a and b-cycles switched). In fact, any closed 3-manifold can be obtained by gluing two handle-bodies by a map of the boundary surface. This decomposition is called a Heegard decomposition of the 3-manifold.

Hatcher 2.2 Problem 35

Suppose there is a an embedding of X in $\mathbb{R}^3$ with a neighborhood U, a mapping cylinder of a map $f : S \to X$, where S is a closed orientable surface. Let $V = S^3 \setminus X$ and write $S^3 = U \cup V$. By the Mayer-Vietoris sequence

$$\cdots \to H_2(S^3) \to H_1(U \cap V) \to H_1(U) \oplus H_1(V) \to H_1(S^3) \to \cdots$$

Since $H_2(S^3) = H_1(S^3) = 1$, this says that there is an isomorphism $H_1(U \cap V) = H_1(U) \oplus H_1(V)$. Note that since U is the mapping cylinder of f, it deformation retracts to X. The set $U \cap V = S \times [0, 1)$ and so deformation retracts to S. So if $H_1(X)$ has torsion we get a contradiction.

Problem 4

(a): The set of 2-chains, $C_2(\Sigma)$ is generated by the triangles T_i, and the 1-chains, $C_1(\Sigma)$ by the edges e_i. Suppose a non-zero element $\sum a_i T_i$ is in the kernel of $\partial : C_2(\Sigma) \to C_1(\Sigma)$. Since each edge is shared by exactly two adjacent triangles, the coefficient a_1 determines completely all other coefficients a_i (in fact, all other coefficients $a_i = \pm a_1$. So the kernel is generated by a single element. Since $C_3(\Sigma) = 0$, the second homology $H_2(\Sigma)$ is isomorphic to the kernel in $C_2(\Sigma)$, so in this case it is $\mathbb{Z}$. The other possibility is that there is no non-zero element in the kernel, in which case $H_2(\Sigma) = 0$.

(b): There is obviously a map of any degree from $T^2 \to T^2$. So it is enough to show that there is a map of degree 1 from $T^2 \to S^2$. The torus T^2 is a square with opposite sides identified. Then, think of S^2 as the square with the entire boundary identified to a single point. This gives the degree 1 map from $T^2 \to S^2$.

On the other hand, let $f : S^2 \to T^2$. Since S^2 is simply connected, f lifts to the universal cover i.e. to a map $F : S^2 \to \mathbb{R}^2$. Since $\mathbb{R}^2$ is contractible, there is a homotopy of f to the constant map. This implies that the degree of f is zero.