HW 2 SOLUTIONS, MA525

1. Problem 2

(a): If p is not onto then the image of p and its complement disconnect X. So we can just focus on the image without changing the problem. Hence assume that p is onto.

Let x be a point in X. We claim that the set of pre-images of x is finite. Assume to the contrary. Since $\widetilde{X}$ is compact and Hausdorff, it is also sequentially compact. So there is at least one accumulation point to the pre-image set. But then p fails to be a local homeomorphism at the accumulation point.

So let $y_1, \cdots, y_n$ be the pre-image set of x. Choose open sets U_i around each y_i such that the restriction of p to each U_i is a local homeomorphism onto $W_i = p(U_i)$. Moreover, since $\widetilde{X}$ is Hausdorff, we may assume that the sets U_i are disjoint. Consider the intersection $W = \cap W_i$, and let $V_i = p^{-1}(W) \cap U_i$. Then the sets V_i evenly cover W by the map p. For each x, we have constructed an evenly covered open set W containing x. Thus the family of sets W cover X, and hence p is a covering map.

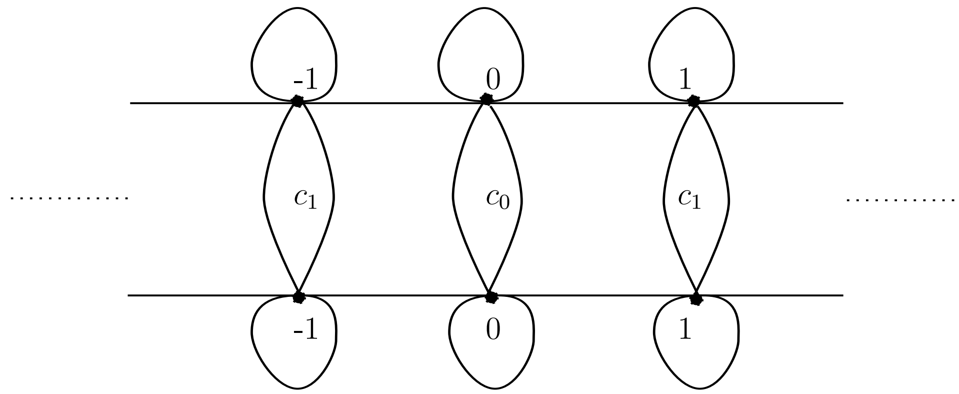
(b): We already showed that the pre-image under p of every point is finite. Suppose now that X is path connected. Let $p^{-1}(x_1)$ denote the pre-image set of x_1. Let $x_2 \in X$ be a point distinct from x_1. Consider some path γ that connects x_1 to x_2. By the path lifting property, given a point in y_1 in $p^{-1}(x_1)$ there is a lift of γ that starts from y_1. Such a lift should necessarily terminate at some point y_2 in $p^{-1}(x_2)$. This gives a map from $p^{-1}(x_1)$ to $p^{-1}(x_2)$.

We claim that this map is injective. Suppose not. Then there are lifts α and β starting from distinct points y_1 and y_1' in $p^{-1}(x_1)$ that terminate at the same point in $p^{-1}(x_2)$. So we can concatenate α to the reverse of β. Denote this path by $\alpha\bar{\beta}$. Then $p(\alpha\bar{\beta}) = \gamma\bar{\gamma}$. The path $\gamma\bar{\gamma}$ can be homotoped while fixing the endpoints to the constant path at x_1. The lift of this homotopy to $\widetilde{X}$ has to deform $\alpha\bar{\beta}$ to a constant path while keeping the endpoints fixed. That is impossible since the endpoints of $\alpha\bar{\beta}$ are distinct points y_1 and y_1'. Thus the claim.

Finally, the claim shows that $\#p^{-1}(x_1) \leq \#p^{-1}(x_2)$. But the argument is completely symmetric in x_1 and x_2. So $\#p^{-1}(x_1) = \#p^{-1}(x_2)$ for distinct points x_1 and x_2.

2. Problem 4

In $\widetilde{X}$, index by $\mathbb{Z}$ the points at which the bouquets of circles are wedged along the horizontal line. Index the circles in each bouquet by the whole numbers. Consider the space Y below



The circles at the dark dots along the two horizontal lines each represent bouquet of circles at each dark dot(too lazy to draw the details). Index the dots on the top line by $\mathbb{Z}$ and similarly the dots on the bottom line by $\mathbb{Z}$ so that they line up pairwise. The space Y admits a map to $\widetilde{X}$ defined as follows:

1. Consider the top horizontal line union the bouquets of circles on it. Map to $\widetilde{X}$ as follows: Send dot i to the corresponding dot of $\widetilde{X}$. Map the bouquet at dot i to the bouquet minus the $|i|$-th circle at dot i in $\widetilde{X}$.
2. Map the bottom horizontal line similar to (1).
3. Map each arc of the bridge between dot i of the top line and dot i of the bottom line to circle $|i|$ at dot i of $\widetilde{X}$ such that it is a homeomorphism restricted to the interior of the arc. Thus circle $|i|$ at dot i is covered twice.

Its obvious that the map above is a two fold cover of $\widetilde{X}$. The composite is not a covering map because it fails to be a covering map at the wedge point in X. The reason is that every open set of the wedge point contains some circle i entirely in it. But no neighborhood of dots $\pm i$ on either of the horizontal lines in Y contains a corresponding circle that maps down.