HW 3 SOLUTIONS, MA525

1. Problem 2

For $t \in [0,1]$, let $f_t : S^1 \times I \to S_1 \times I$ be defined by $f_t(\theta, s) = (\theta + 2\pi st, s)$. Then $f_0 = id$ and $f_1 = f$. Moreover, f_t restricted to $S^1 \times \{0\}$ is the identity map.

Glue $S^1 \times \{0\}$ to $S^1 \times \{1\}$ by $(\theta, 0) \sim (\theta, 1)$ to get a torus T^2. Note that the map f descends down to a map F of the torus. If there is a homotopy f_t between f and id that is identity restricted to $S^1 \times \{0\}$ and $S^1 \times \{1\}$, then it descends down to a homotopy F_t of F to the identity map on the torus. This means that the map F_* induced on $\pi_1(T^2)$ has to be identity.

The path $\gamma_0 : s \to (\theta_0, s)$ maps down to the longitude of the torus. The image $F_*(\gamma_0) \neq \gamma_0$ (in fact, it is the $(1,1)$ curve on the torus). Thus a homotopy as above cannot exist.

2. Problem 3

Let α be a loop based at x_0 in X and let β be a loop based at y_0 in Y. Let δ be the loop in $X \times Y$ given by the concatenation of the loops $\alpha \times \{y_0\}$ and $\{x_0\} \times \beta$ in that order i.e. first $\alpha \times \{y_0\}$ then $\{x_0\} \times \beta$. Let η be the loop in $X \times Y$ given by concatenating in the reverse order.

Consider the 1-parameter family of loops $\gamma_t : [-1, 2] \to X \times Y$ given by

$$\begin{aligned} \gamma_t(s) &= \big(x_0, \beta(t(s+1))\big) & s \in [-1, 0] \\ &= \big(\alpha(s), \beta(t)\big) & s \in [0, 1] \\ &= \big(x_0, \beta((1-t)(s-2)+1)\big) & s \in [1, 2] \end{aligned}$$

In words, the loop γ_t follows β from (x_0, y_0) to $(x_0, \beta(t))$, then does the loop $(\alpha, \beta(t))$ in $X \times \{\beta(t)\}$ and then follows the rest of β from $(x_0, \beta(t))$ back to (x_0, y_0). It is easy to check that this gives a homotopy between δ and η i.e. $\gamma_0 = \delta$ and $\gamma_1 = \eta$.

3. Problem 5

Draw the torus as the square in $\mathbb{R}^2$ with opposite sides identified and with vertices $(\pm 1, \pm 1)$. The union of the longitude and meridian circles of the torus can be taken to be the boundary of this square. Take the origin in $\mathbb{R}^2$ to be the deleted point of the torus. Let $|(x,y)| = \max\{x, y\}$, and for $t \in [0,1]$ let $s = 1 - t(1 - |(x,y)|)$. Consider the deformation retract

$$F_t(x, y) = \frac{1}{s}(x, y)$$

Check that F_0 is the identity on the punctured torus and $F_1(x, y)$ belongs to the boundary of the square.