HW 4 SOLUTIONS, MA525

Problem 5

Suppose X is a tree. Fixing some base-vertex x, for a positive integer k, call all vertices distance k from x as level k vertices. Because of the tree structure, all edges in the tree have endpoints in adjacent levels. We call all edges that have one endpoint in level $(k-1)$ and the other in level k as edges of level k. Again because of the tree structure, every vertex of level k, has exactly one edge of level k terminating on it. This implies that there is a bijection between the vertices of level k and the edges of level k. Taking union over all levels, this tells us that there is a bijection between the set of vertices excluding the base-point and the set of all edges in X. This implies $\chi(X) = 1$.

For a general graph X pick a base vertex x, and let Y be a spanning tree in X from the base-vertex x. Let e be an edge in $X \setminus Y$ with endpoints x_1 and x_2. There is a unique path $[x, x_1]$ in Y from x to x_1 and a unique path $[x, x_2]$ in Y from x to x_2. Identify e with the loop $\gamma(e) = [x, x_1] * e * [x_2, x]$, where $*$ denotes concatenation of paths. Deformation retract the tree Y to the point x, to get a wedge of circles. Each γ_e retracts to a circle in the wedge, and if f is a different edge in $X \setminus Y$, then $\gamma(e)$ and $\gamma(f)$ are distinct circles in the wedge. Thus $\pi_1(X, x)$ is a free group with the basis γ_e. By counting, the number of basis elements is the same as the number of edges in $X \setminus Y$. From the definition of $\chi(X)$, this is the same as $1 - \chi(X)$.

Problem 6

Think of F as $\pi_1(X, x)$ where X is the wedge of k circles at the base-point x, and where each generator a_i of F corresponds to the i-th copy of S^1 in the wedge. We will use the correspondence of covering spaces of X and subgroups of $\pi_1(X, x)$. Let $\widetilde{X}$ be the covering space associated to the subgroup H. Since $n = [F : H]$, the number of pre-images of x_0 is n; denote these by $y_1, \cdots, y_n$.

Consider the lift b_{ik} of the generator a_i starting from y_k. First, note that b_{ik} is an edge from y_k to some y_j i.e. no interior point of b_{ik} can be one of the pre-images. If for some k, the other endpoint of b_{ik} is y_k i.e. the path b_{ik} is a loop at y_k, then for all k, the paths b_{ik} are loops at y_k (because the y_k evenly cover x). Similarly, if for some k, the path b_{ik} is an edge between distinct vertices, then that is true for all k. In particular, a consequence of this is that the cover $\widetilde{X}$ is a finite graph i.e. it has finitely many vertices n each with fixed degree $2k$, where if there is a loop at the vertex it contributes 2 towards the degree.

A finite graph as above is homotopy equivalent to a wedge of circles. This proves that the subgroup H is free and of finite rank. Finally, by Problem 5, the rank of H is $1 - \chi(\widetilde{X})$. The number of vertices in $\widetilde{X}$ is n, and the number of edges in $\widetilde{X}$ is kn, since each vertex is of degree $2k$. So the rank is $(kn - n + 1)$.