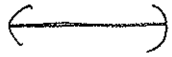
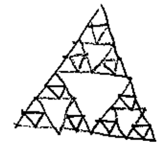
Math 525: Topology — Urbana, Fall 2009

Topological Spaces: Set of points w/ notion of nearness.

pt,  $I=[0,1]$,  $(0,1)$,  Cantor set, , ...

$\mathbb{R}^n$,  circle,  sphere, , S^3, ...

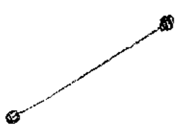
$C(\mathbb{R}^n, \mathbb{R})$, $C^\infty(\mathbb{R}^n, \mathbb{R})$, $\{(x,y)\in\mathbb{C}^2 \mid x^2=y^3+17\}$,

cont fns from $\mathbb{R}^n \to \mathbb{R}$

[configuration space of a robot arm...]

Metric Space: (X, d) dist fn sat: symmetry, Δ inequality, non degenerate. [Query]

Eg. $X=\mathbb{R}^2$, $d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$


$\vec{x}$, $\vec{y}$

From this define

① Convergence of a sequence
② Continuity of functions.
③ Closed and open sets.


ε, x

using open balls

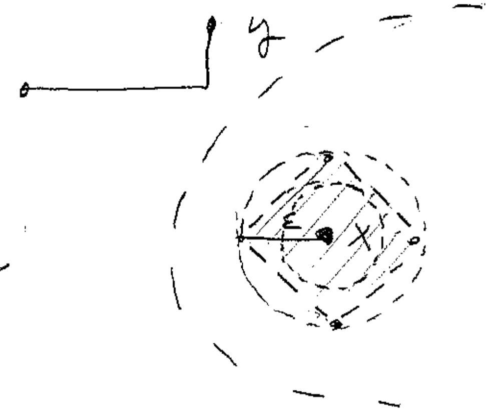
$$B_\varepsilon(x)=\{y\in X \mid d(x,y)<\varepsilon\}$$

$U \subseteq X$ is open if $\forall x \in X$ $\exists \varepsilon > 0$ such that

$$B_\varepsilon(x) \subseteq U.$$

Some other metrics on $\mathbb{R}^2$ define the same open sets, e.g.

$$d'(x,y) = |x_1 - y_1| + |x_2 - y_2|$$



In geometry the focus is on the metric; in topology its on the open sets.

Def: A topological space is a set X with a collection $\mathcal{U}$ of subsets sat:

open sets

1. $\emptyset$ and X are in $\mathcal{U}$.
2. $U_1, U_2 \in \mathcal{U} \Rightarrow U_1 \cap U_2 \in \mathcal{U}$.
3. $U_\alpha \in \mathcal{U} \Rightarrow \bigcup_\alpha U_\alpha \in \mathcal{U}$.

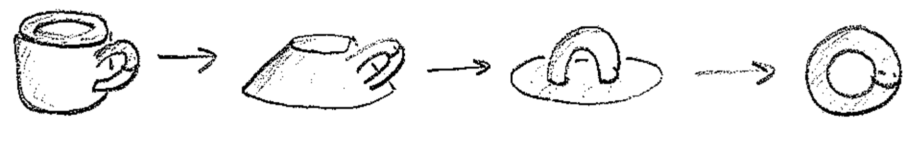
Ex: X a metric sp. $\mathcal{U}$ = usual open sets.

[Most of the examples we'll focus on in this class are actually metric spaces.]

Def: $f: X \to Y$ is continuous if $\forall$ open $U \subseteq Y$ $f^{-1}(U)$ is open in X.

Top spaces X and Y are homeomorphic if there is a bijection $f: X \to Y$ where f and f^{-1} are both continuous.

Problem: Given X, Y are they homeomorphic?

Yes: Give homeo: 

No: Need a property to distinguish them.

Easy: $(0,1) \not\cong [0,1]$ Compactness

$\mathbb{R} \not\cong \mathbb{R}^2$ Connectedness of space − pt.

Harder: $\mathbb{R}^2 \not\cong \mathbb{R}^3$,  $\not\cong$ .

[Need additional prop/invariants. An imp source of such is:]

Algebraic Topology:

$$X \rightsquigarrow F(X)$$

some algebraic object e.g. an abelian group

[top sp]

[Depends only on the homeo type of X]

Also

$$X \xrightarrow[\text{cont fn}]{f} Y \text{ gives } F(X) \xrightarrow{f_*} F(Y)$$

Respects the alg str.

Brower Fixed Point Thm: $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$

If $f: D^n \to D^n$ is cont, then $\exists x \in D^n$ with $f(x) = x$.

Geometric Topology: Study of spaces that are locally nice, e.g. manifolds.

Course in a nutshell:

Invariants of algebraic topology as applied to the examples of geometric topology.

Discuss syllabus, etc: