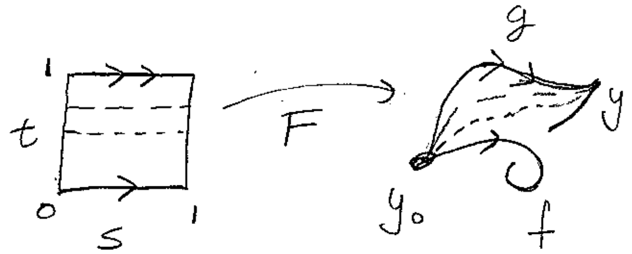
Lecture 4:

Last time:



Path homotopy
$F(s,0) = f(s)$
$F(s,0) = g(s)$
$F(0,t) = y_0$
$F(0,t) = y_1$

$\boxed{f \simeq_p g}$

$[f]$: path equivalence class of f.

$\pi_1(Y, y_0) = \{ [f] \mid f \text{ path starting + ending at } y_0 \}$

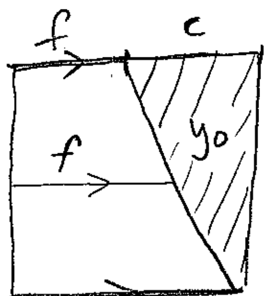
Prop: The $\circ$ operation makes $\pi_1(Y, y_0)$ into a group

Ident: If c is the constant path at y_0,
then $[f]\cdot[c] = [c]\cdot[f] = [f] \quad \forall [f]$.

Inverses: $[f]^{-1} = [\bar{f}]$ where $\bar{f}(s) = f(1-s)$.

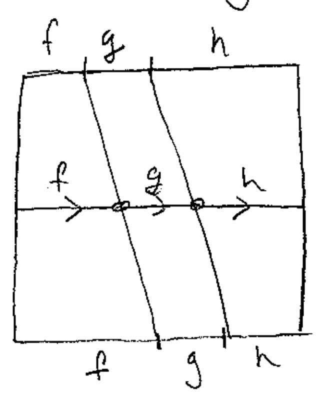
Assoc: $([f]\cdot[g])\cdot[h] = [f]\cdot([g]\cdot[h])$

Pf: Ident: $f \circ c \simeq_p f$ via

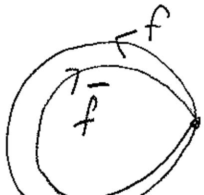


$$F(s,t) = \begin{cases} f\left(\frac{s}{1-t/2}\right) & \text{if } s \le \frac{t}{2} \\ y_0 & \text{otherwise} \end{cases}$$

Assoc:



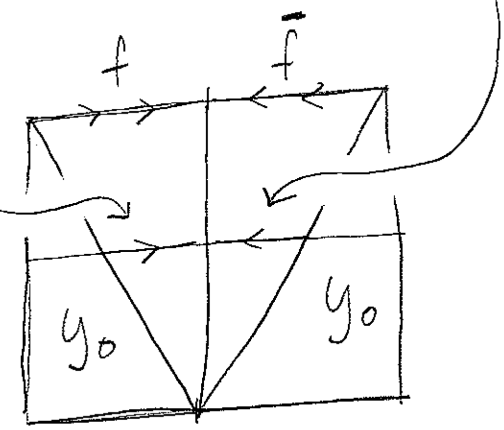
Inverses



$[f]\circ[\bar f] =$  $= \bullet$

Last 1/2 of $\bar f$

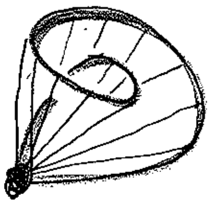
First 1/2 of f



Ex: $\pi_1(\mathbb{R}^n, \vec{0}) = 1$

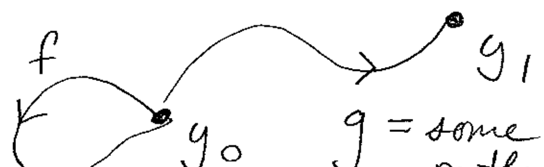
Pf: Any path f is $\simeq$ to the const path at $\vec{0}$ via

$$F(s,t) = t\, f(s)$$



[Note: Same for any pt in $\mathbb{R}^n$.]

Prop: If y_0 and y_1 can be joined by a path, then $\pi_1(Y, y_0) \cong \pi_1(Y, y_1)$.



Pf: Use $G: [f] \longmapsto [\bar g \cdot f \cdot g]$

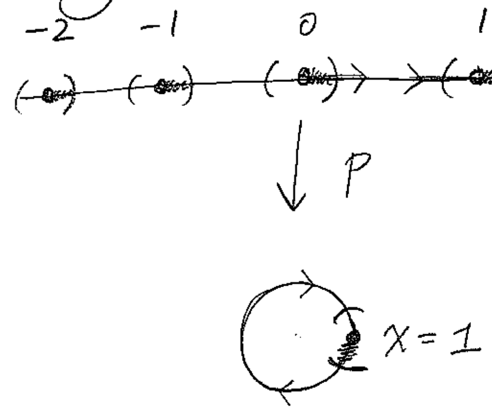
This is a homomorphism as

$$
\begin{aligned}
G([f])\,G([f']) &= [\bar g \cdot f \cdot g]\cdot[\bar g \cdot f' \cdot g] \\
&= [\bar g \circ f]\circ\underbrace{[g \circ \bar g]}_{c_{y_0}}\cdot[f' \circ g] \\
&= [\bar g \circ f \circ f' \circ g] = G([f]\circ[f'])
\end{aligned}
$$

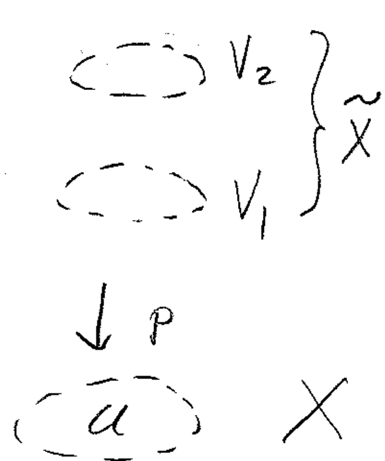
Moreover, G^{-1} is just $[f'] \longmapsto [g \circ f' \circ \bar g]$ ∎

[To compute a nontrivial π_1, need a new tool...]

<u>Covering Spaces</u>: $p : \mathbb{R} \to S^1 \quad p(t) = e^{-2\pi t i}$
$= (\cos 2\pi t, -\sin 2\pi t)$

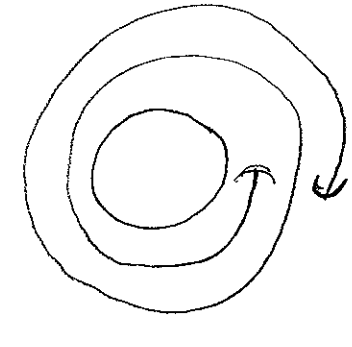


<u>Def</u>: A map $p : \tilde{X} \to X$ <u>evenly</u> <u>covers</u> $U \overset{\text{open}}{\subseteq} X$ if $p^{-1}(U) = \underset{\text{union}}{\text{disjoint}}\ V_\alpha \overset{\text{open}}{\subseteq} \tilde{X}$ where $p|_{V_\alpha} : V_\alpha \to U$ is a homeo for all α.



<u>Def</u>: $p : \tilde{X} \to X$ is a <u>covering</u> <u>map</u> iff every $x \in X$ has a nbhd which is evenly covered. ($\tilde{X}$ is called a covering space of X.)

<u>Ex</u>: $p : \mathbb{R} \to S^1$ above.



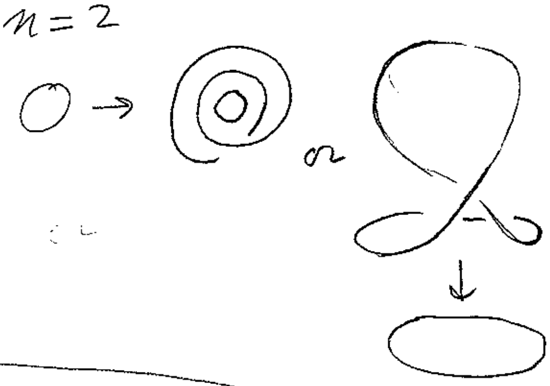
<u>Non Ex</u>: Not enough that p is a local homeo, e.g. $p|_{(0,2)}$

More examples:

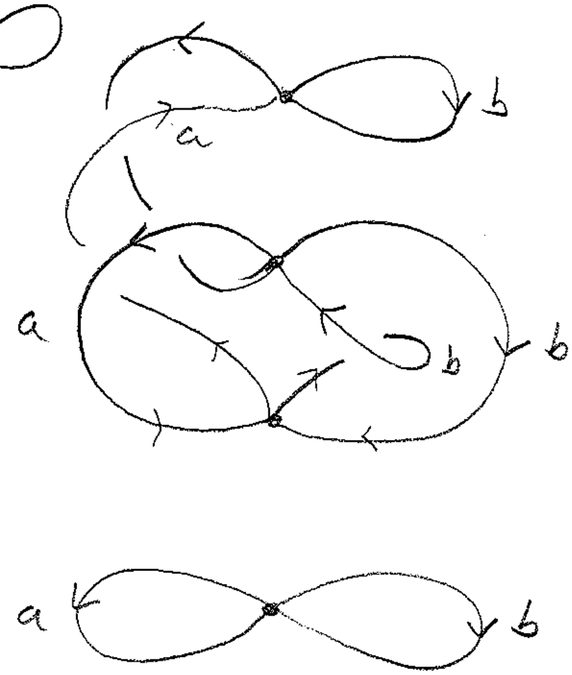
$p: S^1 \longrightarrow S^1$

$z \longmapsto z^n$

$n = 2$



$X =$ 



Many more examples on page 58.