Lecture 35: The formal viewpoint (§2.3)

[How many diff homologies are there? At least one for each G...]

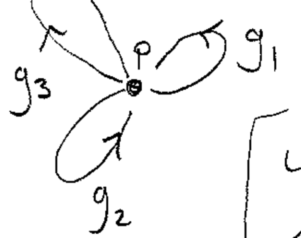
Category: $\mathcal{C}$ consist of

1) A collection of objects $Ob(\mathcal{C})$.
2) $\forall X, Y \in Ob(\mathcal{C})$ have $Mor(X,Y)$, the morphisms.
   Special $1_X \in Mor(X,X)$, the "identity".
3) Composition of morphisms: $\forall X, Y, Z \in Ob(\mathcal{C})$
   have $\circ : Mor(X,Y) \times Mor(Y,Z) \longrightarrow Mor(X,Z)$.
   Sat $(f \circ g) \circ h = f \circ (g \circ h)$ and $f \circ 1 = 1 \circ f = f$.

Ex: Category of Top. Spaces: $Ob =$ all top spaces
$Mor(X,Y) =$ cont fns $f: X \to Y$.

Category of Abelian Groups: $Mor(X,Y) =$ homomorphisms $X \to Y$.

G a group. $Ob = \{p\}$ $Mor(p,p) =$ elts of G.
$\circ =$ group mult



[A category w/ one object is a "group w/o inverses", i.e. a monoid. A category where every morphism is invertable is a groupoid.]

A <u>Functor</u> from $\mathcal{C}$ to $\mathcal{D}$ is $X \in Ob(\mathcal{C}) \longmapsto F(X) \in Ob(\mathcal{D})$

which satisfies: $f \in Mor(X,Y) \longmapsto Mor(F(X), F(Y))$.

where $X, Y \in Ob(\mathcal{C})$.

$F(1_X) = 1_{F(X)}$

$F(f \circ g) = F(f) \circ F(g)$

<u>Ex</u>: ① Singular Homology $H_n : \begin{pmatrix} \text{cat of} \\ \text{top sps} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{cat of} \\ \text{ab. gps} \end{pmatrix}$

$X \longmapsto H_n(X; \mathbb{Z})$

② Abelianization: $\begin{pmatrix} \text{cat of} \\ \text{gps} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{cat of} \\ \text{abelian gps} \end{pmatrix}$

③ π_1 is a functor from $\begin{pmatrix} \text{cat. of top} \\ \text{spaces with} \\ \text{basepoints} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{cat of} \\ \text{gps} \end{pmatrix}$

[These are covariant functors; there are also contravariant ones. Categories were created to describe alg. top., but now use many places (ie C.S.).]

<u>Natural Transformation</u>: $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ functors

$\forall x \in Ob(\mathcal{C})$ have $T_X : F(X) \longrightarrow G(X)$ (a morphism)

such that $\forall f \in Mor(X,Y)$ then

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle T_X}\downarrow & & \downarrow{\scriptstyle T_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array}$$ commutes.

Ex: A, B abelian gps. $F, G : \mathcal{T}op \to AbGroups$

$$F(X) = H_n(X; A)$$
$$G(X) = H_n(X; B)$$

If $\varphi : A \to B$ is a hom, then $T_X = \varphi_*$ from $H_n(X;A) \to H_n(X;B)$ is a nat'l transformation. [We used this prop last time to understand degrees of map of spheres.]

A reduced homology theory is:

A functor $\tilde{h} : \begin{pmatrix} \text{CW complex} \\ \text{and cont. maps} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{sequences of} \\ \text{abelian gps} \end{pmatrix}$

$$X \longmapsto \{\tilde{h}_n(X)\}_{n \in \mathbb{Z}}$$

satisfying

① $f \simeq g : X \to Y$ then $\tilde{h}(f) = \tilde{h}(g)$

② $A \subseteq X$ a subcomplex. Have two functors:

$$\begin{pmatrix} \text{CW complexes} \\ \text{w/ a subcomplex} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{seq of} \\ \text{ab. gps} \end{pmatrix}$$

$$(X,A) \longmapsto \{\tilde{h}_n(X/A)\}$$

$$(X,A) \longmapsto \{\tilde{h}_{n-1}(A)\}$$

There exists a nat'l trans

$$\partial : \tilde{h}_n(X/A) \longrightarrow \tilde{h}_{n-1}(A)$$

such that

$$\xrightarrow{\partial} \tilde{h}_n(A) \xrightarrow[\tilde{h}(i)]{} \tilde{h}_n(X) \xrightarrow[\tilde{h}(q)]{} \tilde{h}_n(X/A) \xrightarrow{\partial} \tilde{h}_{n-1}(A) \to$$

is exact.

③ $X = \bigvee_\alpha X_\alpha$ then $\bigoplus_\alpha \tilde{h}(X_\alpha) \xrightarrow{\bigoplus_\alpha \tilde{h}(i_\alpha)} \tilde{h}(X)$

is an isomorphism.

<u>Ex</u>: Fix G, consider $X \longmapsto \{\tilde{h}_n(X) = \tilde{H}_n(X;G)\}$

<u>Ex</u>: K-theory, bordism.

Thm: Suppose $\tilde{h}$ is a reduced homology theory of CW complexes. If $\exists G$ s.t.

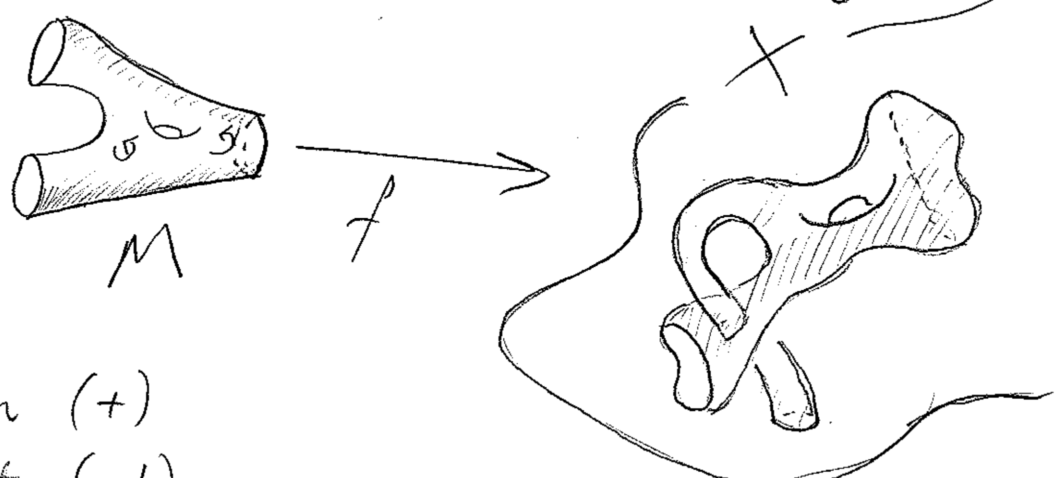
$$\tilde{h}_n(S^0) = \begin{cases} G & \text{when } n=0 \\ 0 & \text{otherwise} \end{cases}$$

then $\tilde{h}_n(X) = \tilde{H}_n(X; G)$

[Pf: Very similar to the proof that simplicial and singular homology are the same. Also think of our calc of $H_n(S^k; G)$ yesterday, and the matter of degree.]

Bordism: X space.

$$C_n^{\Omega}(X) = \left\{ \begin{array}{l} f: M \to X \text{ where} \\ M \text{ is a smooth oriented compact} \\ n\text{-manifold, poss. w/ boundary} \end{array} \right\}$$



Opperations:

disjoint union $(+)$
change orient (-1)
$\{\emptyset\}$ is zero. This a free abelian gp.

$$C_n^{\Omega}(X) \xrightarrow{\partial} C_n^{\Omega}(X)$$

$$(f : M \to X) \longmapsto (f|_{\partial M} \to X)$$

$\partial^2 = 0 \Rightarrow H_n^{\Omega}(X)$ = bordism homology.

For small n (≤ 3), $H_n^{\Omega}(X) \cong H_n(X)$

<u>Note</u>: Really $C_n^{\Omega}(X)$ is the free ab. gp on $(f : M \to X)$ + certain relations.