Lecture 38: Final applications

Last time:

Thm: a) $h: D^k \hookrightarrow S^n$ then $\tilde{H}_i(S^n \setminus h(D^k)) = 0 \ \forall i$.

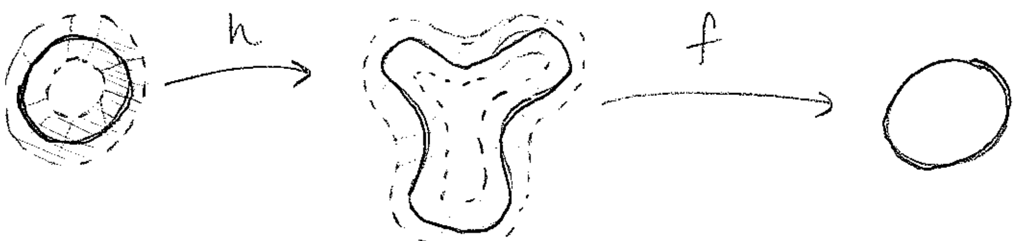
b) $h: S^k \hookrightarrow S^n$ for $k < n$. Then

$$\tilde{H}_i(S^n \setminus h(S^k)) = \begin{cases} \mathbb{Z} & \text{for } i = n-k-1 \\ 0 & \text{otherwise.} \end{cases}$$

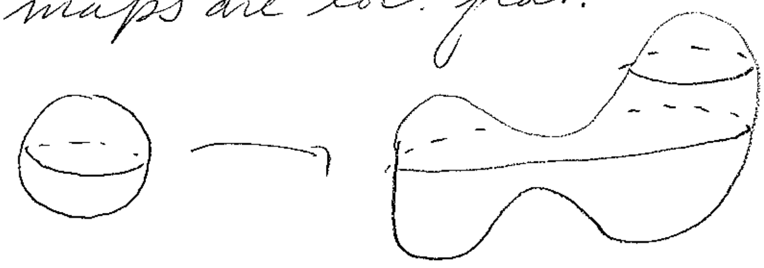
Cor: $h: S^n \hookrightarrow S^{n+1}$. The complement of $h(S^n)$ has two connected components.

Saw that $\exists$ non-standard (wild) $h: D^k, S^k \hookrightarrow S^n$.
In the case of the Cor, wildness is purely local:

Thm (Mazur-Brown 1960s) $h: S^n \hookrightarrow S^{n+1}$ locally flat, i.e. extends to an embedding $S^n \times I \hookrightarrow S^{n+1}$.
Then $\exists f: S^{n+1} \circlearrowleft \cong$ with $f(h(S^n)) = S^n$.



Ex: Smooth maps are loc. flat.



The fact that the codim is 1 is key to [MB].

Here's the images of three different smooth maps

$S^1 \hookrightarrow S^3$: 

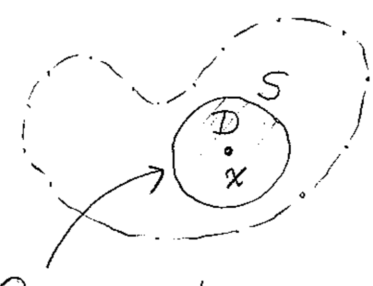
$\pi_1(S^3 \setminus h(S^1)) = \mathbb{Z}$ $\quad \langle x,y \mid x^2=y^3 \rangle \quad \langle x,y \mid x^3 y x^{-1} y^{-2} x^{-1} y = 1 \rangle$

[Aside: Knot theory from Kelvin to DNA...]

Sim, there are smoothly knotted $S^2 \hookrightarrow S^4$, but not $S^1 \hookrightarrow S^4$!

<u>Invariance</u> of <u>Domain</u>: Suppose $X \subseteq \mathbb{R}^n$ is homeo to an open set in $\mathbb{R}^n$. Then X is open.

<u>Pf</u>: Can replace $\mathbb{R}^n$ with S^n. Any $x \in X$ is contained in $D \cong D^n$, and let S corresp to ∂D^n under this homeo. Now $S^n \setminus S$ is open and has two connected comp by Thm (b).



As

$$S^n \setminus S = \underbrace{S^n \setminus D}_{\text{conn by Thm(a)}} \amalg \underbrace{D \setminus S}_{\text{conn as} \cong \text{open ball}}$$

<u>Goal</u>: Show $D \setminus S$ is open in S^n!

$\Rightarrow$ these are the two components $\Rightarrow$ they are open $\Rightarrow$ $D \setminus S$ is open in S^n $\Rightarrow$ X is open. ▨

<u>Cor</u>: M a cpt n-mfld, N a connected n-mfld. Any embedding $M \hookrightarrow N$ is onto.

<u>Cor^2</u>: S^n does not embed in $\mathbb{R}^n$

<u>Cor^3</u>: $\mathbb{R}^n$ does not contain a subspace $\cong \mathbb{R}^k$ for $k > n$.

<u>Pf</u>: If $\mathbb{R}^n \supseteq \mathbb{R}^k$ it contains S^n, a contradiction.

<u>Pf of Cor</u>: M is closed in N as the former is cpt and the latter Hausdorff. Each $x \in M$ is contained in $U \subseteq M$, open in M, and $V \subseteq N$, open in N, where $U \cong V \cong \mathbb{R}^n$. Can assume $U \subseteq V \Rightarrow U$ is open in $V \Rightarrow U$ is open in N $\Rightarrow M$ is open. As M is also closed, must have $M = N$. ▨

Now for some 21^{st} century mathematics!

In 1904, Poincaré proposed

Conj: If M is a cpt 3-mfld with $\pi_1 = 1$, then $M \cong S^3$.

Note, analog true for surfaces though not 4-manifolds ($S^2 \times S^2$). Proved by Perelman in 2003 using ideas of Hamilton (Ricci Flow) and Thurston (Geom = Top in dim 3.

Fun fact: Poincaré originally suggested $H_1(M; \mathbb{Z}) = 0 \Rightarrow M \cong S^3$ but he later realized that this was false.
($S^3/_{\text{binary icosohedral group}}$)

Conj: M a cpt n-mfld. If $M \simeq_{h.e.} S^n$ then $M \cong S^n$.

Proved by Smale in 1960s for $n \geq 5$
Freedman in 1980s for $n = 4$.

Blather about this and that if time remains.

The End