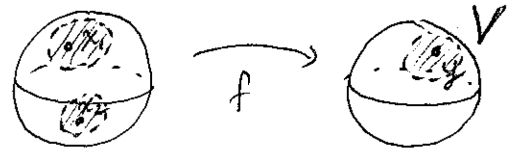
Lecture 31: Applications of degree



Last time: Suppose $y \in S^n$ has $f^{-1}(y) = \{x_i\}$ finite.
V a nbhd of y, U_i disjoint nbhds of x_i w/ $f(U_i) \subseteq V$.

$$\bar{\alpha}_i \longmapsto (\deg_{x_i} f)\,\bar{\alpha}$$

$$\begin{tikzcd}
 & H_n(U_i, U_i \setminus x_i) \arrow[r, "f_*"] \arrow[d, "k_i"] \arrow[dl, "\cong"'] & H_n(V, V \setminus y) \arrow[d, "\cong"] \\
H_n(S^n, S^n \setminus x_i) & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n \setminus y) \\
 & H_n(S^n) \arrow[u, "j_*"] \arrow[ul, "\cong"] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

$\bar{\alpha}_i'$ $\qquad\qquad$ $\bar{\alpha}'$

$$\alpha \longmapsto (\deg f)\,\alpha$$

$$H_n(S^n, S^n \setminus f^{-1}(y)) \cong \bigoplus_i \mathbb{Z} \quad \text{gen by } k_i(\bar{\alpha}_i) = \bar{\alpha}_i'$$

If $j_*(\alpha) = \sum a_i \bar{\alpha}_i'$, then $p_i(j_*(\alpha)) = a_i = 1$.

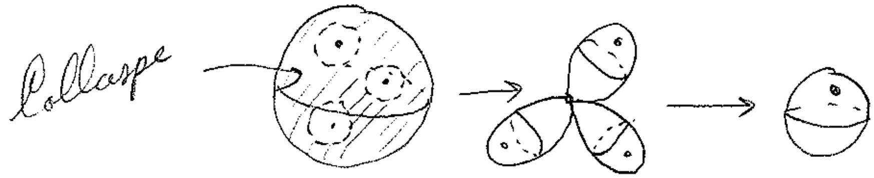
So $j_*(\alpha) = \sum \bar{\alpha}_i'$. Then $f_*(j_*(\alpha)) = \sum f_*(\bar{\alpha}_i')$

$= \sum (\deg_{x_i} f)\,\bar{\alpha}'$. Thus

$$\boxed{\deg f = \sum_i \deg_{x_i} f.}$$

Cor: Given $k \in \mathbb{Z}$, $\exists f: S^n \to S^n$ of deg k.

Pf: Consider $S^n \longrightarrow \bigvee_{|k|} S^n \longrightarrow S^n$ map each wedge summand by a homeo of deg = sign(k).



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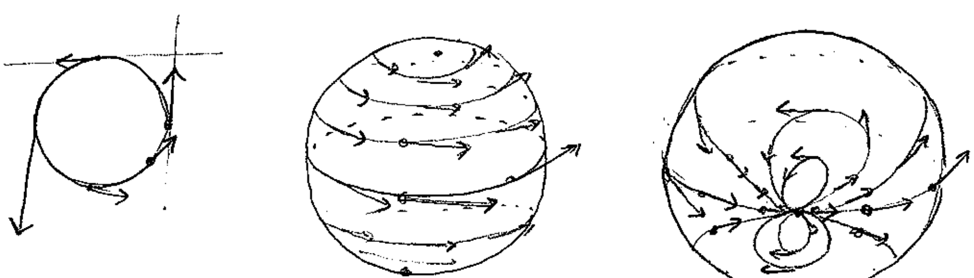
Vector Fields: $V: S^n \longrightarrow \mathbb{R}^{n+1}$ where $V(x) \in T_x S^n$.

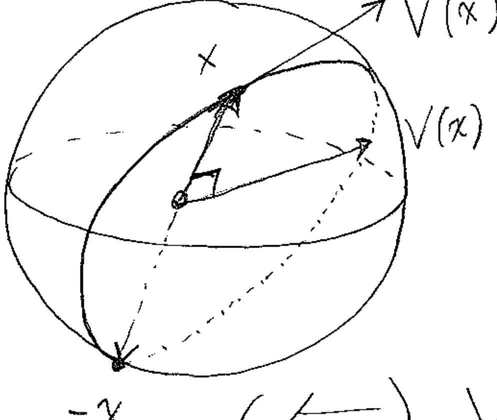
Ex:



Thm: S^n has a nowhere vanishing vector field $\iff$ n is odd.

Proof: ($\Rightarrow$) Suppose V is nowhere vanishing. Rescale V so it has unit length $\left(= \dfrac{V(x)}{|V(x)|}\right)$. Consider

$f_t: S^n \times [0,1] \longrightarrow S^n$ given by $f_t(x) = \cos(\pi t)\, x + \sin(\pi t)\, V(x)$



Then $f_0 = id$ is hom. to $f_t = A$.

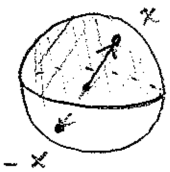
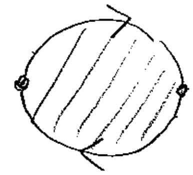
So $1 = \deg(id) = \deg(A) = (-1)^{n+1}$

Hence n is odd.

($\Leftarrow$) $V(x_1, \ldots, x_{n+1}) = (-x_2, x_1, -x_4, x_3, \ldots, -x_{n+1}, x_n)$

$\blacksquare$

Q: What spaces is S^n the univ cover for? $(n \geq 2)$

Ex: $\mathbb{R}P^2 = S^2/_{x \mapsto -x} =$  $=$  $\Rightarrow \pi_1 = \mathbb{Z}/2$ in two ways.

$\mathbb{R}P^n = S^n/_{x \mapsto -x}$

Ex: Lens spaces: $n \geq 1$, $a \in [1, n-1]$ rel. prime to n.

$\zeta_n = e^{2\pi i/n}$ $\quad \mathbb{Z}/n$ acts on $S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$

by $(z,w) \longmapsto (\zeta_n z, \zeta_n w)$

then:
$S^3 \longrightarrow S^3/_{\mathbb{Z}/n} = L_{a/n}$ is a covering map. [Q: Why? A: free, finite]

So: $\pi_1(L_{a/n}) = \mathbb{Z}/n$. Saw $L_{1/n}$ on HW.

[Same construction works in any odd dimension.]

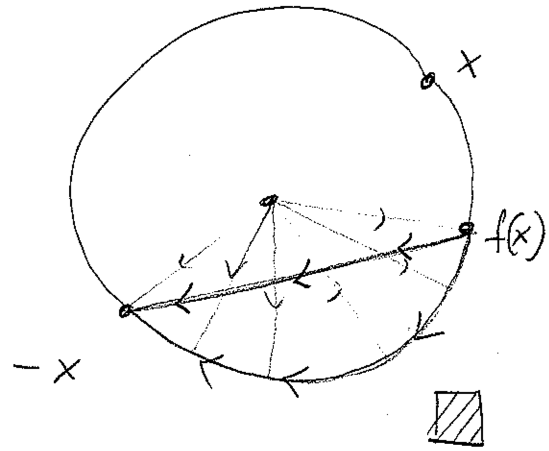
Thm: $G \neq 1$ a finite group acting freely on S^n.
If n is even, then $G \cong \mathbb{Z}/2$.

[Note: $\mathbb{Z}/2$ does act via antipodal map.]

Proof: Consider $d: G \to \{\pm 1\}$, $g \longmapsto \deg(g)$. This is a hom to $\{\pm 1\} \cong \mathbb{Z}/2$ as $\deg(f \circ g) = \deg(f)\deg(g)$.

Claim: If $S^n \circlearrowleft f$ has no fixed points, then $f \simeq A$.

Pf from claim: Since n is even, the claim implies every $g \neq 1$ in G has $\deg = -1$. So $\ker d = 1$, and d is an isomorphism.



Pf of claim: $f_0 = A$, $f_1 = f(x)$ when

$$f_t(x) = \frac{(1-t)(-x) + t f(x)}{\|(1-t)(-x) + t f(x)\|}$$

Q: $f : S^n \circlearrowleft$ with no fixed pts, $f^2 = 1$. Does there exist $g : S^n \circlearrowleft$ s.t. $g^{-1} \circ f \circ g = A$ $\left(\iff S^n/_f = \mathbb{RP}^n\right)$?

$\underline{n=1}$: yes $\underline{n=2}$: yes by class. of surfaces. $\underline{n=3}$: yes [Livesay 1960]

$\underline{n \geq 4}$: no!

Thm (Perelman 2003): $S^3/\mathbb{Z}/n$ is always a lens space.

He also proved

Poincaré Conj: M^3 cpt 3-mfld (w/o boundary)

If $\pi_1 M = 1$, then $M^3 \cong S^3$.

Note: The following is false:

~~Conj~~: M^3 a non cpt contractible 3-mfld. Then $M^3 \cong \mathbb{R}^3$

[Cf. Extra credit prob on MT 1.]

If time remains:

1. Review the class. of surfaces
2. Poincaré in high dims. E.g. $\pi_1(S^2 \times S^2) = 1$.

Next week: Joan Nevins

No class Friday.