Lecture 34: Homology w/ Coefficients

$G =$ abelian group: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}/2\mathbb{Z}$, etc. [opperation written additively]

$$C_n(X;G) = \bigoplus_{\substack{\text{sing.}\\ \text{simp}}} G = \left\{ \sum_{\text{finite}} g_\alpha \sigma_\alpha \;\middle|\; \begin{array}{l} g_\alpha \in G \\ \sigma_\alpha \text{ sing. simplex} \end{array} \right\}$$

$$= C_n(X;G) = C_n(X) \otimes G$$

$$\partial(g\sigma) = \sum (-1)^k g\, \sigma\Big|_{k^{th}\text{ face}}$$

- makes sense even when G is <u>not</u> a ring: $n \cdot g = \underbrace{g + \cdots + g}_{n \text{ times if } n>0.}$
- $\partial^2 = 0$, etc.

Resulting

$H_n(X;G)$ has all the usual properties

[long exact seq, excision, cellular version.]

<u>Ex</u>: $\tilde{H}_k(S^n;G) = \begin{cases} G & n = k \\ 0 & \text{otherwise} \end{cases}$

<u>$n=0$</u>: $\xrightarrow{0} C_2(S^0;G) \xrightarrow{\cong} C_1(S^0;G) \xrightarrow{0} C_0(S^0;G) \xrightarrow{\varepsilon} G \to 0$

takes each σ_0^k to 1_G

$C_0(S^0;G) \cong G \oplus G$

$H_0 = \ker \varepsilon = \{(g,-g)\} \cong G.$

<u>Note</u>: Exactly two singular n-cells σ_n^1 and σ_n^2 in each dim.

$\underbrace{\bullet \qquad \bullet}_{S^0}$

<u>$n>0$</u>: Use the long exact sequence

$$\underset{0}{\tilde{H}_k(D^{n+1};G)} \to H_{k+1}(D^{n+1},\partial D^{n+1};G) \xrightarrow{\cong} \tilde{H}_k(\partial D^{n+1};G) \to \tilde{H}_k(D^{n+1};G)$$

$H_{k+1}(D^{n+1},\partial D^{n+1};G) \cong \tilde{H}_{k+1}(D^{n+1}/\partial D^{n+1} \cong S^{n+1};G)$

$\partial D^{n+1} \cong S^n$

$\tilde{H}_k(D^{n+1};G) = 0$

Cellular Homology: $C_n^{CW}(X;G) = H_n(X^n, X^{n-1}; G)$

$d_n = j \circ \partial_n$ as before $\qquad = \bigoplus_{n\text{-cells}} G$

elts of g

<u>Lemma</u>: For an n-cell e_α, set $d_n(g e_\alpha) = \sum_{\substack{n-1 \\ \text{cells} \\ e_\beta}} c^g_{\alpha\beta}\, e_\beta$.

Then $c^g_{\alpha\beta} = c_{\alpha\beta}\, g$ where

$c_{\alpha\beta}$ = degree of attaching map of e_α to image of e_β in $X^{n-1}/X^{n-2} = \bigvee_\beta S^{n-1}$. [i.e. $c_{\alpha\beta}$ is from the case $G = \mathbb{Z}$.]

This follows directly from:

<u>Lemma</u>: $f: S^n \circlearrowleft$ of degree d. Then $f_*: \tilde{H}_n(S^n;G) \circlearrowleft$ is mult by d (i.e. $x \longmapsto d.x$)

Pf: In general, if $\varphi: G \to H$ is a hom. of ab. gps, then get $\varphi_*: H_n(X;G) \to H_n(X;H)$.

Let $c \in \tilde{H}_n(S^n;G)$, and g be the corr elt of G.

[<u>Well def</u>: Once we fix the ident of $D^{n+1}/\partial D^{n+1} \cong S^{n+1}$, the only choice is the ident of $\ker \varepsilon$ with G which is fixed once we choose an ordering of the two pts of S^0.]

Consider $\varphi: \mathbb{Z} \longrightarrow G$, $1 \longmapsto g$. Then $\varphi_*\left(\begin{smallmatrix}\text{gen } \alpha \\ \text{of } H_n(S^n;\mathbb{Z})\end{smallmatrix}\right) = c$

by induction, using the long exact seq.

Then

$$\begin{tikzcd} & \alpha \arrow[r, mapsto] & (\deg f)\alpha & \\ \mathbb{Z} \cong & \tilde{H}_n(S^n;\mathbb{Z}) \arrow[r, "f_*"] \arrow[d, "\varphi_*"'] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{H}_n(S^n;\mathbb{Z}) \arrow[d, "\varphi_*"] & \cong \mathbb{Z} \\ G \cong & \tilde{H}_n(S^n;G) \arrow[r] & \tilde{H}_n(S^n;G) & \cong G \\ & c \arrow[r, mapsto, "f_*"] & (\deg f)c. & \end{tikzcd}$$

$\blacksquare$

$\mathbb{R}P^n$: CW chains w/ $G = \mathbb{Z}$:

$$\cdots \xrightarrow{\times 2} \underset{3}{\mathbb{Z}} \xrightarrow{0} \underset{2}{\mathbb{Z}} \xrightarrow{\times 2} \underset{1}{\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}}$$

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } n=k \text{ is odd, or } k=0, \\ \mathbb{Z}/2 & \text{if } 0<k<n \text{ is odd.} \end{cases}$$

$G = (\mathbb{R},+)$:

$$0 \longrightarrow \underset{n}{\mathbb{R}} \xrightarrow{\times 2 \text{ or } 0} \;\; \to \mathbb{R} \xrightarrow{0} \underset{2}{\mathbb{R}} \xrightarrow{\times 2} \underset{1}{\mathbb{R}} \xrightarrow{0} \underset{0}{\mathbb{R}} \longrightarrow 0$$

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{R} & \text{if } n=k \text{ and } n \text{ is odd or } k=0. \\ 0 & \text{otherwise.} \end{cases}$$

$G = (\mathbb{F} = \mathbb{Z}/2\mathbb{Z}, +)$

since $\times 2 = 0$ on $\mathbb{F}$

$$0 \longrightarrow \mathbb{F} \xrightarrow[\substack{\times 2 \text{ or } 0 \\ = 0}]{} \mathbb{F} \to \cdots \to \mathbb{F} \xrightarrow[0]{} \mathbb{F} \xrightarrow[0]{} \mathbb{F} \longrightarrow 0$$

$$H_k(\mathbb{R}P^n;\mathbb{F}) = \begin{cases} \mathbb{F} & \text{for } 0<k\leq n \\ 0 & \text{otherwise} \end{cases}$$

Fact: $H_n(X;G)$ is completely det. by $H_n(X;\mathbb{Z})$ and $H_{n-1}(X;\mathbb{Z})$.

For $G = \mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, if

$$H_n(X;\mathbb{Z}) = \mathbb{Z}^k \oplus \Big(\bigoplus \mathbb{Z}/\underbrace{d_i}_{\text{odd}}\mathbb{Z}\Big) \oplus \Big(\bigoplus_{i=1}^{j} \mathbb{Z}/\underbrace{e_i}_{\text{even}}\mathbb{Z}\Big)$$

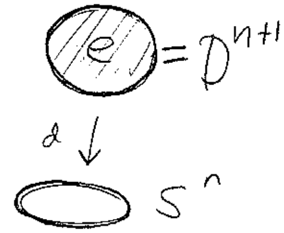
and

$$H_{n-1}(X;\mathbb{Z}) = \mathbb{Z}^{k'} \oplus \Big(\quad " \quad \Big) \oplus \Big(\bigoplus_{i=1}^{j'} \mathbb{Z}/e_i'\mathbb{Z}\Big)$$

then $H_n(G;\mathbb{F}) = \mathbb{F}^{k+j+j'}$. See §3.A for details.

[Still useful, esp. field coeffs. "One prime at a time."]

Moore Space: $X = S^n$ w/ one $(n+1)$-cell attached by a map of degree d:



Consider $X \xrightarrow{q} X/S^n = S^{n+1}$ Is this map null-homotopic?

$$\begin{array}{lccccccc}
 & & n+1 & & n & & & 0 \\
\mathbb{Z}\text{-homology:} & 0 \longrightarrow & \mathbb{Z} & \xrightarrow{\times d} & \mathbb{Z} & \longrightarrow 0 & \cdots \longrightarrow & \mathbb{Z} \\
\tilde{H}_n(X) & 0 & 0 & & \mathbb{Z}/d\mathbb{Z} & 0 & \cdots & 0
\end{array}$$

So any map $q: X \to S^{n+1}$ is 0 on $\tilde{H}_*$

$\underline{\mathbb{Z}/d\mathbb{Z}\text{-homology}}$: $H_{n+1}(X; \mathbb{Z}/d\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$

and $q_* : H_{n+1}(X; \mathbb{Z}/d\mathbb{Z}) \xrightarrow{\cong} H_{n+1}(S^{n+1}; \mathbb{Z}/d\mathbb{Z})$

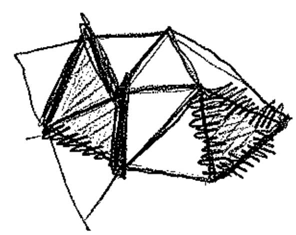
clear on the cell level.

So q is <u>not</u> null homotopic.

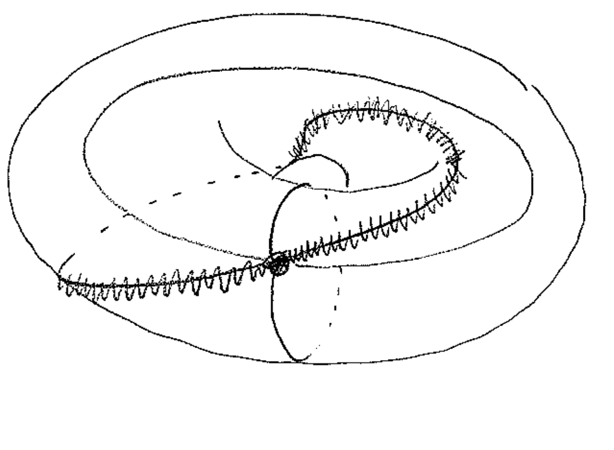
$\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ coeffs for a finite Δ-complex X.

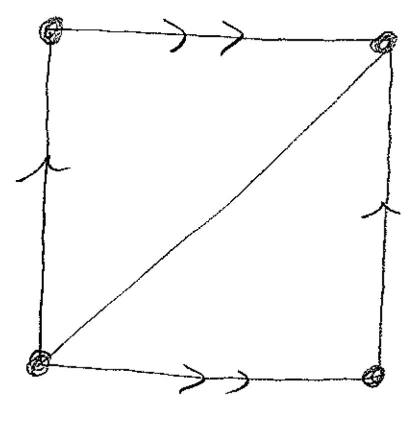
$$C_n(X; \mathbb{F}) = \left\{ \sum_{n \text{ cells}} c_\alpha e_\alpha \;\middle|\; c_\alpha \in \mathbb{F} \right\} = \left\{ \begin{array}{l} \text{subsets of the} \\ n\text{-cells} \end{array} \right\}$$

∂c = all $(n-1)$-cells which are faces of an <u>odd</u> number n-cells in c.



$\partial^2 = 0$: clear ("look ma, no signs.")





$H_0(T; \mathbb{F}) = \mathbb{F}$ $\quad$ $H_1(T; \mathbb{F}) = \mathbb{F}^2$ $\quad$ $H_2(T; \mathbb{F}) = \mathbb{F}$

non-zero elts given by the 3-loops

gen is both triangles.