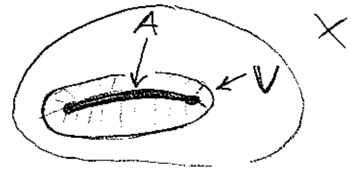
Lecture 27: Excision



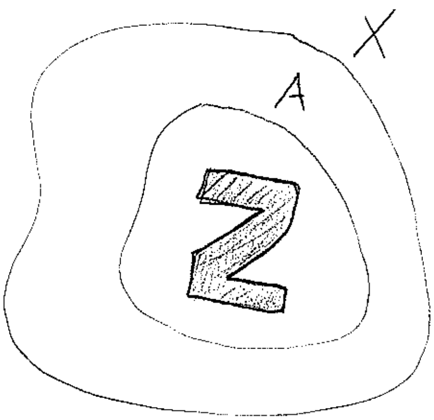
Goal: Good Pair: $A^{\text{clsd}} \subseteq X$ has a nbhd V which def. retracts to it

Thm: (X,A) a good pair. Then the quotient map $q: (X,A) \to (X/A, A/A)$ induces an isom

$$H_n(X,A) \xrightarrow{q_*} H_n(X/A, A/A) \cong \tilde{H}_n(X/A) \text{ for all } n.$$

[Combines to give the long exact seq of the pair.]

Excision: $Z \subseteq A \subseteq X$, with $\overline{Z} \subseteq \text{int}(A)$. ← largest open set $\subseteq A$.



Then the map

inclusion

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{i_*} H_n(X,A)$$

is an isomorphism.

[Motivate, Jump up and down about, etc.]

Pf of Thm from Excision: In general, $H_n(Y, pt) \cong \tilde{H}_n(Y)$ since

$$\to \underset{0}{H_n(pt)} \to H_n(Y) \xrightarrow{\cong} H_n(Y, pt) \to \underset{0}{H_{n-1}(pt)} \to$$

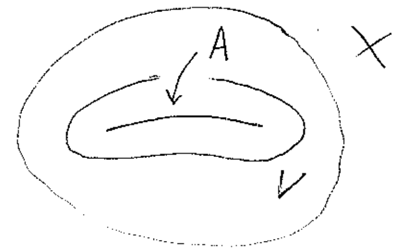
and

Clearly injective

$$\begin{tikzcd} H_1(pt) \arrow[r] & H_1(Y) \arrow[r, "\cong"'] & H_1(Y,pt) \arrow[r] \arrow[dr] & H_0(pt) \arrow[r] & H_0(Y) \arrow[r] & H_0(Y,pt) \arrow[r] & 0 \\ 0 & & & 0 \arrow[u] & & H_0(Y)/\mathbb{Z} \arrow[u, equal] & \end{tikzcd}$$

with $H_0(pt) = \mathbb{Z}$.

<u>Claim</u>: $H_n(X,A) \xrightarrow{i_*} H_n(X,V)$



<u>Pf</u>: Long exact sequence of the triple

$$\to \underset{0}{H_n(V,A)} \to H_n(X,A) \xrightarrow[\cong]{i_*} H_n(X,V) \xrightarrow{\partial} \underset{0}{H_{n-1}(V,A)} \to$$

comes from the short exact sequence

$$0 \to C_*(V,A) \to C_*(X,A) \to C_*(X,V) \to 0$$

of chain complexes.

$$\begin{tikzcd} H_n(X,A) \arrow[r, "i_*", "\cong"'] \arrow[d, "q_*"'] \arrow[dr, phantom, "\circlearrowleft"] & H_n(X,V) \arrow[d, "q_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(X \setminus A, V \setminus A) \arrow[l, "i_*\ \cong"', "\text{by Excision}"] \arrow[d, "q_*", "\cong \text{ as is a homeo}"'] \\ H_n(X/A, A/A) \arrow[r] & H_n(X/A, V/A) & H_n(X/A \setminus A/A, V/A \setminus A/A) \arrow[l, "\cong"', "\text{by Excision}"] \end{tikzcd}$$

$\cong$ By long exact seq of the triple

Commuting of the two squares forces leftmost g_* to be an $\cong$. $\blacksquare$

[Because of missed classes, won't prove Excision in detail.]

<u>Setup:</u> $\mathcal{U} = \{U_i\}$ with $\bigcup \operatorname{int}(U_i) = X$.

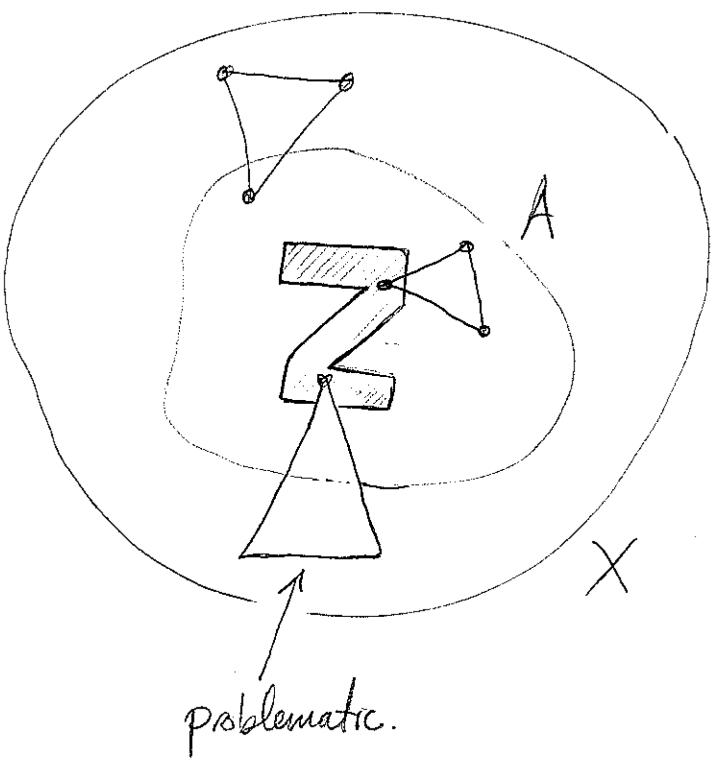
Set $C_n^{\mathcal{U}}(X) \leq C_n(X)$ gen by $\sigma : \Delta^n \to X$ w/ image $\subseteq$ some U_i.

A subcomplex, so have $H_n^{\mathcal{U}}(X)$.

For excision, take $\mathcal{U} = \{A, B = X - Z\}$

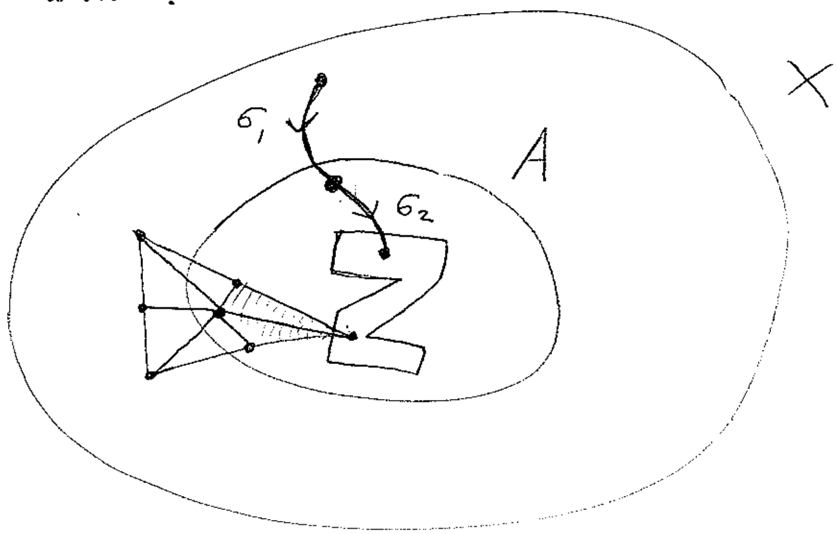
<u>Reason:</u> $C_n(X \setminus Z, A \setminus Z) \xrightarrow{i} C_n(X, A)$

Want a map back, but only maps sense on $C_n^{\mathcal{U}}(X)$.



Prop: $i : C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$ is a chain homotopy equivalence, $\exists \rho : C_*(X) \to C_*^{\mathcal{U}}(X)$ s.t. $\rho \circ i$ and $i \circ \rho$ are chain homotopic to the ident. In particular,

$H_n^{\mathcal{U}}(X) \to H_n(X)$ is an isom.

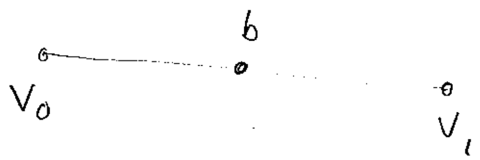


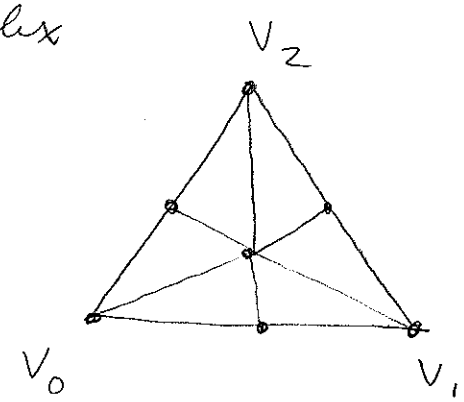
Constructing ρ:

Barycentric Subdivision:

$\delta = [v_0, v_1, \ldots, v_n]$ an n-simplex

Barycenter $= \frac{1}{n+1} \sum v_i = b$

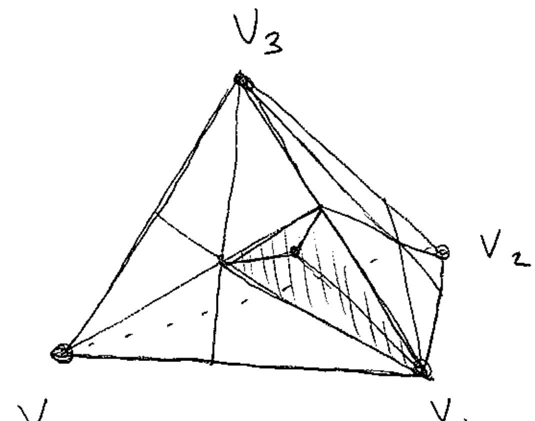




Subdivision Op:

$S : C_n(X) \longrightarrow C_n(X)$

$$\sigma \longmapsto \sum_{\substack{\tau \text{ in barycentric} \\ \text{subdiv. of } \Delta^n}} \pm \sigma|_\tau$$



A chain map chain hom. to the ident.

Rough idea: $\rho = S^n$, but not quite as the # of times we need to subdivide σ so that it lies in $C_n^{\mathcal{U}}(X)$ depends on σ. See Hatcher for details.

Next time: $H_n^{\Delta}(X) \cong H_n(X)$.

Then onto applications!