Lecture 16: Lifting again

Goal: X path conn, loc. path conn. Then two covering spaces are isomorphic iff

$$\boxed{p_2 \circ f = p_1} \qquad \begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

$p_*(\pi_1 \tilde{X}_1)$ and $p_*(\pi_1 \tilde{X}_2)$ are conjugate in $\pi_1 X$.

[For reasonable spaces, this means covers are classified by subgps of $\pi_1 X$.]

Basic Question:

When can we lift f?

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p\text{-covering map}"] \\ (Y, y_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & (X, x_0) \end{tikzcd}$$

$\tilde{f}$ satisfies

$p \circ \tilde{f} = f$

$\tilde{f}(y_0) = \tilde{x}_0$.

[Always can when $Y \cong I^n$.]

Key: If $\tilde{f}$ exists, then

$$f_*(\pi_1(Y, y_0)) = (p \circ \tilde{f})_*(\pi_1(Y, y_0))$$
$$= p_*(\tilde{f}_*(\pi_1(Y, y_0)))$$

Hence

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

<u>Thm</u>: Y path conn, loc. path conn.
Then f lifts to $\tilde{f}: Y \to \tilde{X}$ with
$\tilde{f}(y_0) = \tilde{x}_0$ if and only if
$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

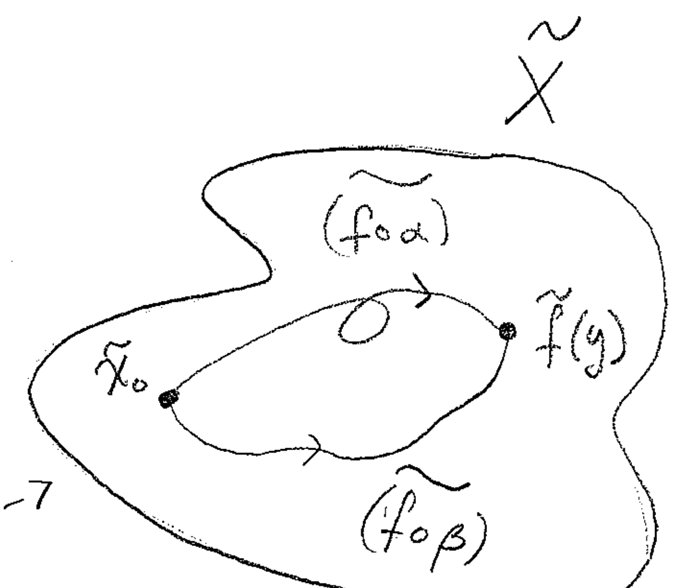
covering sp.

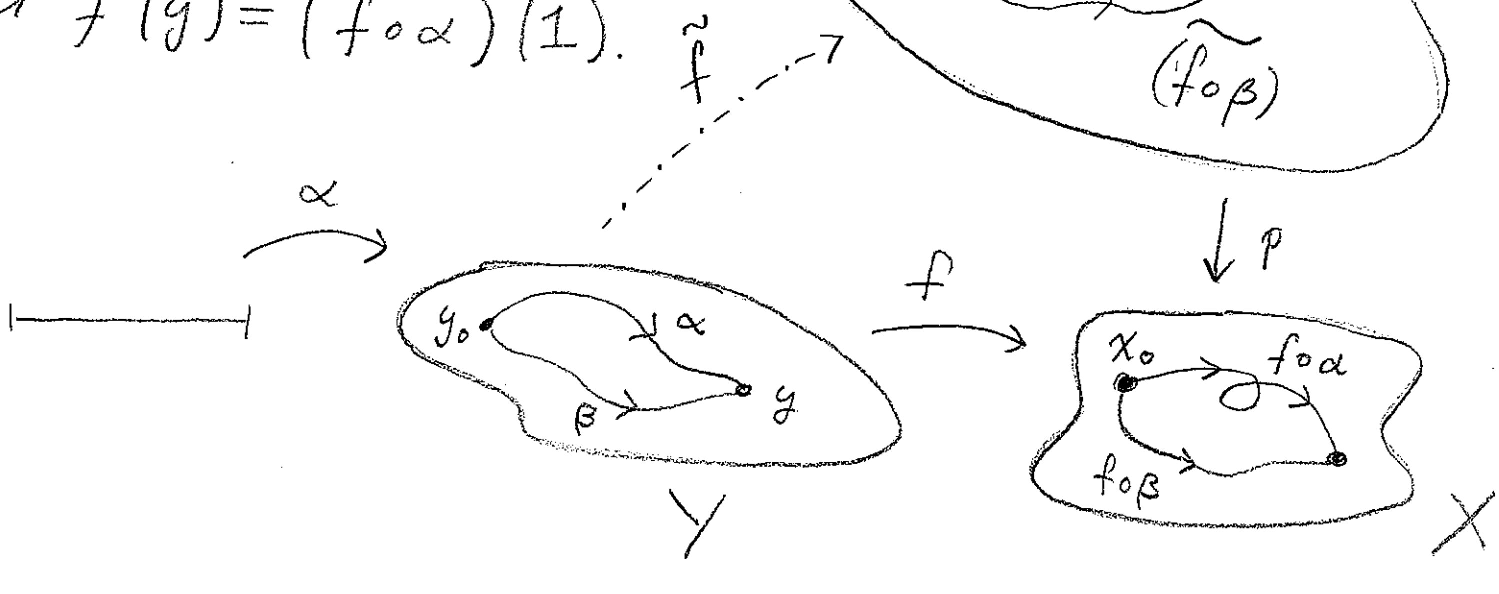
$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0) \end{tikzcd}$$

[Classic model application of alg. top.]

<u>Proof</u>: ($\Rightarrow$) See above.

($\Leftarrow$) For $y \in Y$, choose a path α from y_0 to y.
Set $\tilde{f}(y) = (\widetilde{f \circ \alpha})(1)$.

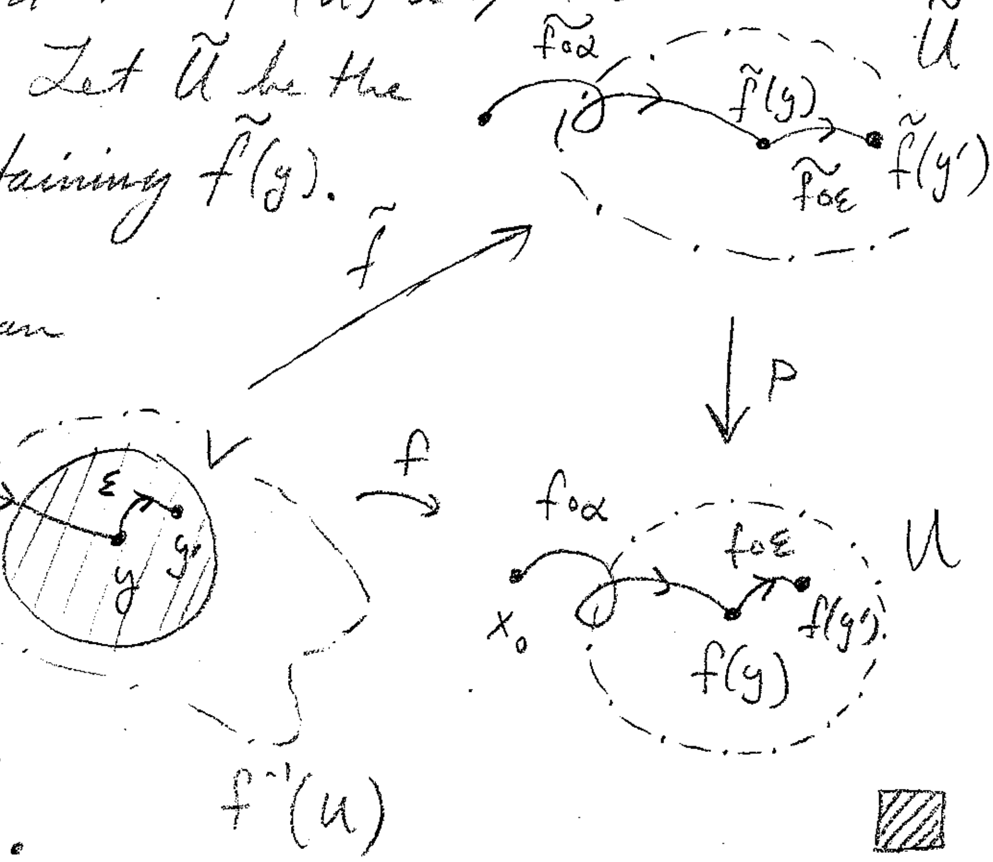




<u>Point</u>: <u>If</u> $\tilde{f}$ exists, then $\tilde{f}\circ\alpha$ is a lift of $f\circ\alpha$ starting at $\tilde{x}_0$. By uniqueness of path lifts, must have $\tilde{f}\circ\alpha = \widetilde{(f\circ\alpha)}$.

<u>Claim</u>: indep of α. Let β be another path from y_0 to y. By hyp. $f_*([\alpha\cdot\bar{\beta}]) \subseteq p_*(\pi_1(\tilde{X},\tilde{x}_0))$ i.e. $(f\circ\alpha)\cdot\overline{(f\circ\beta)}$ lifts to $\tilde{X}$ as a <u>loop</u> at $\tilde{x}_0$. Thus $\widetilde{f\circ\alpha}(1) = \widetilde{f\circ\beta}(1)$. [Also notice that $\tilde{f}$ is clearly a lift.]

<u>Claim</u>: $\tilde{f}$ is cont. Let U be an evenly covered nbhd of $f(y)$, and $V \subseteq f^{-1}(U)$ a path conn open nbhd of y. Let $\tilde{U}$ be the slice above U containing $\tilde{f}(y)$.



For any y' in V, can define $\tilde{f}$ via $\alpha\cdot\varepsilon$ where $\varepsilon \subseteq V$.

Thus $\tilde{f}|_V = (P|_{\tilde{U}})^{-1}\circ f$.

Hence $\tilde{f}$ is continuous.

Addendum: When the lift exists, it is unique.

Thm: Covering spaces of a path conn, loc path conn space X are isomorphic via an f taking $\tilde{x}_1$ to $\tilde{x}_2$

$$\begin{tikzcd}
(\tilde{X}_1, \tilde{x}_1) \arrow[rr, dashed, "f"] \arrow[dr, "p_1"'] & & (\tilde{X}_2, \tilde{x}_2) \arrow[dl, "p_2"] \\
& (X, x_0) &
\end{tikzcd}$$

iff $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$

Pf: ($\Rightarrow$) Clear

($\Leftarrow$) Regarding p_2 as the covering map, the last thm gives that p_1 lifts to $f: (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ with $p_1 = p_2 \circ f$. Sim, $\exists\, g: (\tilde{X}_2, \tilde{x}_2) \longrightarrow (\tilde{X}_1, \tilde{x}_1)$ with $p_2 = p_1 \circ g$. Now consider

$$\begin{tikzcd}
(\tilde{X}_1, \tilde{x}_1) \arrow[rr, "g \circ f"] \arrow[dr, "p_1"'] & & (\tilde{X}_1, \tilde{x}_1) \arrow[dl, "p_1"] \\
& (X, x_0) &
\end{tikzcd}$$

and note $p_1 \circ (g \circ f) = p_1$

since $(p_1 \circ g) \circ f = p_2 \circ f = p_1$. Thus $g \circ f$ is a lift of p_1. Another lift is $id_{\tilde{X}_1}$, so have $g \circ f = id_{\tilde{X}_1}$ ∎

Same for $f \circ g = id_{\tilde{X}_2}$ so $f + g$ are inv. isomorphisms