Lecture 20: Homology 101

Reminder: Exam Friday

Note: Can bring 1 sheet of paper to the exam.

π_1: fine as far as it goes, but need "higher dim'l" invariants. [Can't tell S^2 from S^3, det by $X^{(2)}$ etc.]

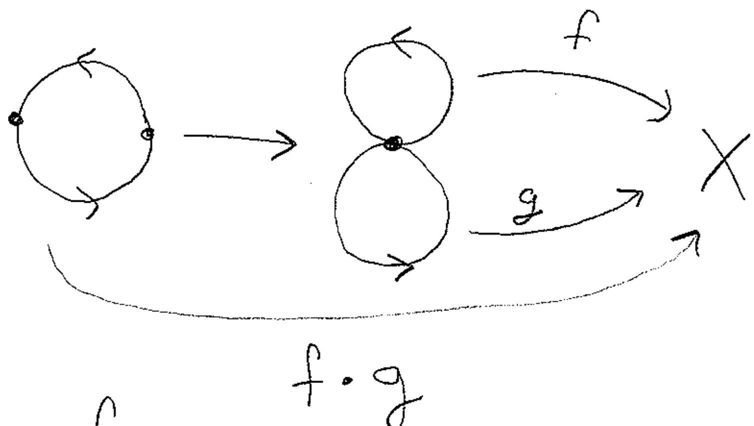
Higher homotopy groups:

$\pi_n(X, x_0)$ = homotopy classes of maps

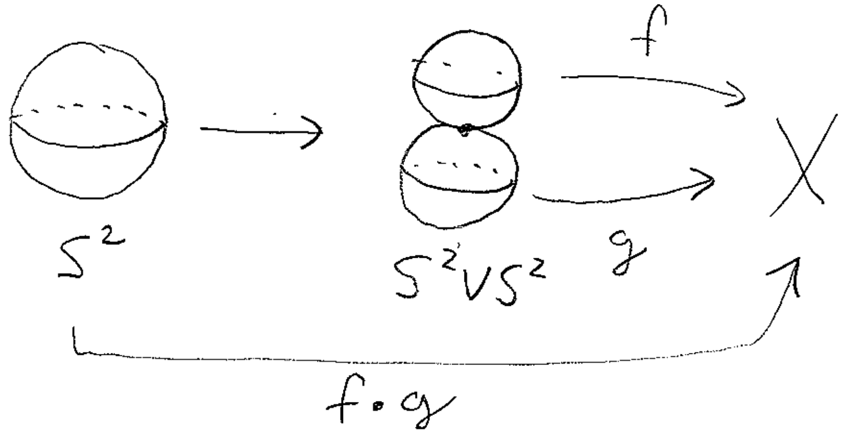
$$(S^n, s_0) \longrightarrow (X, x_0)$$

[homotopies pres. basept.]

Operation: for π_1:



for π_2:



[Same for π_n.]

[Like π_1, the group π_n depends only on X up to homotopy.]

<u>Problem</u>: Hard to compute

[the flip side is that it contains a lot of info about the space]

$\pi_1 S^2 = 1$

$\pi_2 S^2 = \mathbb{Z}$ (gen by id)

$\pi_3 S^2 = \mathbb{Z}$ (Hopf fibration)

$\pi_4 S^2 = \mathbb{Z}/2$

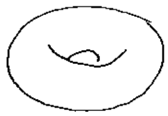
$\pi_5 S^2 = \mathbb{Z}/2$

$\pi_6 S^2 = \mathbb{Z}/12$

$\vdots$

$\exists$ an algorithm to compute $\pi_n S^m$

Another problem with π_1 is that its hard to tell if two finitely-presented groups are the same.

<u>Ex</u>:  $\not\cong$ 

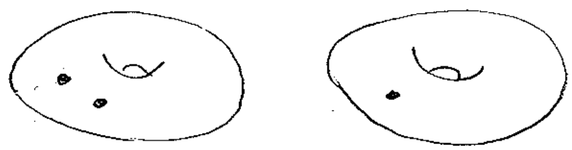
π_1: $\langle a,b \mid aba^{-1}b^{-1} = 1\rangle$ $\quad$ $\langle a,b,c,d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1\rangle$

$\pi_1^{ab} = \pi_1 / [\pi_1, \pi_1] = \mathbb{Z}^2$ vs. $\mathbb{Z}^4$!

Homology:

$$H_n(X) = \frac{\text{n dim'l things w/o boundary}}{\text{boundaries of n+1 dim'l things}}$$

Ex: $H_0(X) = \mathbb{Z}^{(\# \text{ of path comps})}$

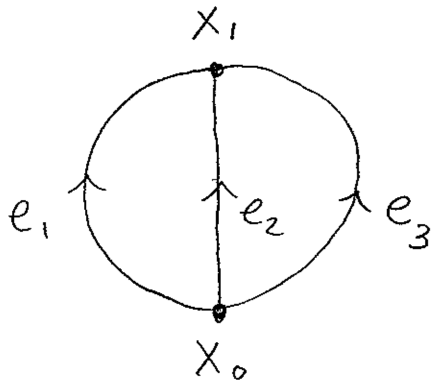
$X =$ 

typical elt of H_0

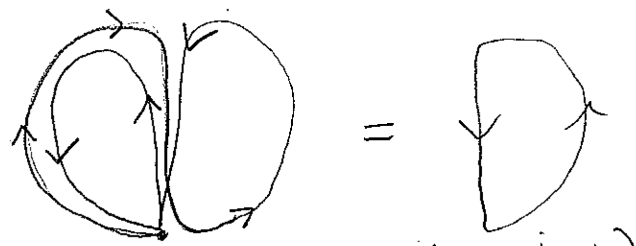
two elts which are equal

Points have "signs": $\overset{+1}{\circ}\ \overset{+1}{\circ} = \overset{+2}{\circ} \Rightarrow H_0(X) = \mathbb{Z}^2.$

Ex: $H_1(X) = \pi_1(X)^{ab}$

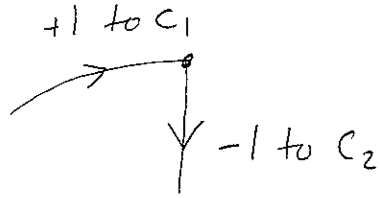
$X =$ 

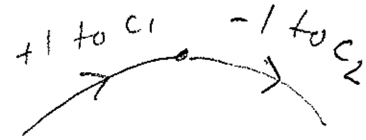
$aba^{-1} \in \pi_1 X = b$



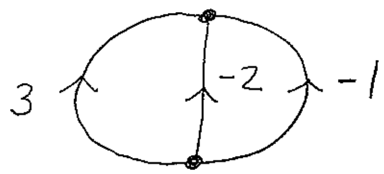
after abelianization

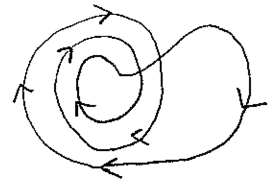
$\gamma \in \pi_1 X$ get #s c_1, c_2, c_3 corr to how many times we cross each edge with sign

Not all tripples appear 

or 

Thus $c_1 + c_2 + c_3 = 0$.

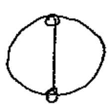
This is sufficient, too 



Thus $\pi_1^{ab} = \text{tripples } (c_1, c_2, c_3) \big/ {c_1+c_2+c_3=0} = \mathbb{Z}^3 \big/ {c_1+c_2+c_3=0} \cong \mathbb{Z}^2$

X a space with a cell decomp.

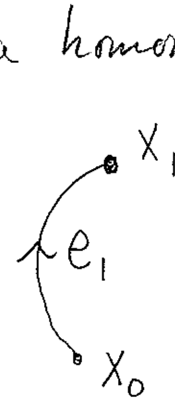
<u>n-chains:</u> $C_n(X)$ = free abelian gp gen by the n-cells.

<u>Ex</u>: $X =$ 

$$C_0(X) = \mathbb{Z} \oplus \mathbb{Z} = \{a_0 x_0 + a_1 x_1\}$$
$$C_1(X) = \mathbb{Z}^3 = \{c_1 e_1 + c_2 e_2 + c_3 e_3\}$$
$$C_n(X) = 0 \text{ for } n > 1.$$

<u>Boundary map:</u> $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ a homomorphism.

$$\partial_1 : C_1(X) \longrightarrow C_0(X) \qquad \partial_1(e_1) = x_1 - x_0$$
$$\partial_1(e_i) = x_1 - x_0$$



<u>Cycles</u> (things w/o boundaries)

$c \in C_n(X)$ with $\partial c = 0$, i.e. $\ker \partial_n$

<u>0-cycles:</u> $\ker \partial_0 = C_0(X)$.

<u>1-cycles:</u> $\partial_1(c_1 e_1 + c_2 e_2 + c_3 e_3) =$

$$c_1(x_1 - x_0) + c_2(x_1 - x_0) + c_3(x_1 - x_0)$$
$$= (c_1 + c_2 + c_3)\, x_1 - (c_1 + c_2 + c_3)\, x_0$$

$\Rightarrow \ker \partial_1 =$ those with $c_1 + c_2 + c_3 = 0$.

$$\underline{H_n(X)} = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

change of basis

$$H_0(X) = C_0(X) / c(x_1 - x_0) = \mathbb{Z}^2 / (1,-1) = \mathbb{Z}$$

$$H_1(X) = \ker \partial_1 / \operatorname{im} \partial_2 = \ker \partial_1 = \left\{ \begin{matrix} \text{those} \\ \text{with} \end{matrix}\ c_1 + c_2 + c_3 = 0 \right\} = \mathbb{Z}^2$$

with basis $e_1 - e_2$,
$e_2 - e_3$

More complicated:

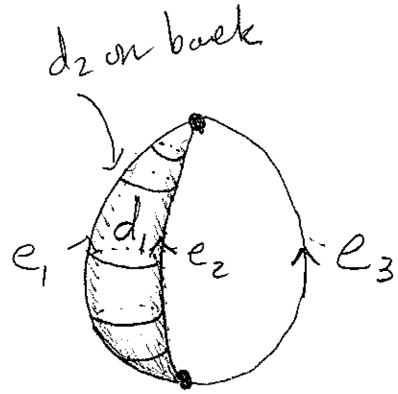
$$C_2(X) = \mathbb{Z} \text{ gen by } d_1$$

$$\partial_2(d_1) = e_2 - e_1$$

$$H_1(X) = {}^{\ker \partial_1}\!/\!_{\operatorname{im} \partial_2} = \mathbb{Z} \text{ gen by } e_2 - e_3$$

$$H_2(X) = 0 \quad \text{as } \ker \partial_2 = 0.$$

Finally:



$H_0 = \mathbb{Z}$

$H_1 = \mathbb{Z}$

$H_2 = \mathbb{Z}$ gen by $d_1 - d_2$

$H_n = 0 \quad n > 2.$