Lecture 25: Long exact sequences.

Goal: $A^{\text{clsd}} \subseteq X$. Want to relate $H_*(A)$, $H_*(X)$, $H_*(X/A)$.

Homology: $\to C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \to 0$

$$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

Here $\varepsilon(\text{sing. 0-simplex}) = \underline{1}$ and say why $\varepsilon \circ \partial_1 = 0$

Reduced Homology:

$$\to C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon = \partial_0} \mathbb{Z} \to 0$$

$$\tilde{H}_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

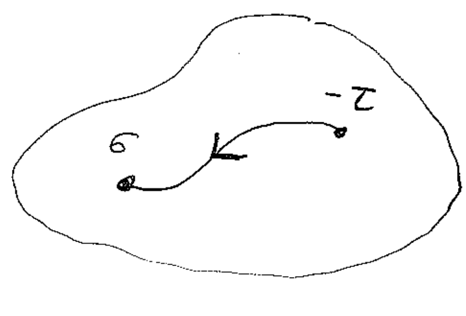
[$= H_n(X)$ when $n > 0$
<u>smaller</u> than $H_0(X)$.]

[Technically conv., simplifies "bookkeeping".]

What is $\tilde{H}_0(X)$ when X is path conn?

$$\ker \varepsilon = \left\{ \sum a_i \sigma_i \;\middle|\; \sum a_i = 0 \right\}$$

$$= \left\langle \sigma - \tau \;\middle|\; \begin{array}{c}\sigma, \tau \text{ sing} \\ \text{0-simps}\end{array} \right\rangle$$



So: $\ker \varepsilon \subseteq \operatorname{im} \partial_1 \Rightarrow \tilde{H}_0(X) = 0$.

In general, $\tilde{H}_0(X) \oplus \mathbb{Z} \cong H_0(X)$.

Note: X contractible $\Rightarrow \tilde{H}_n(X) = 0$ for all n.

Exact sequences:

groups

$$\longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}}$$

where $\ker \alpha_n = \operatorname{im} \alpha_{n+1}$ for all n.

[Q: Have people seen this before? How does it differ from a chain complex.]

Ex: $0 \longrightarrow A \xrightarrow{\alpha} B$ is exact $\Longleftrightarrow$ α is 1-1.

$B \xrightarrow{\beta} C \rightarrow 0$ is exact $\Longleftrightarrow$ β is onto.

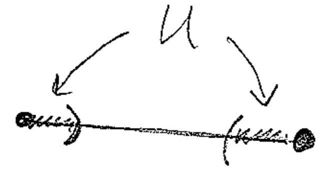
So

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \quad \text{is exact}$$

means that B has a subgp isom to A with quotient gp C. ["Short exact sequence"]

<u>Good Pair</u>: $A \subseteq X$ with A clsd and having a nbhd that def. retracts to A.

<u>Ex</u>: $X = I$, $A = \partial I = \{0, 1\}$



<u>Ex</u>: X a CW complex, A a subcomplex.

<u>Thm</u>: (X,A) a good pair. Then the sequence

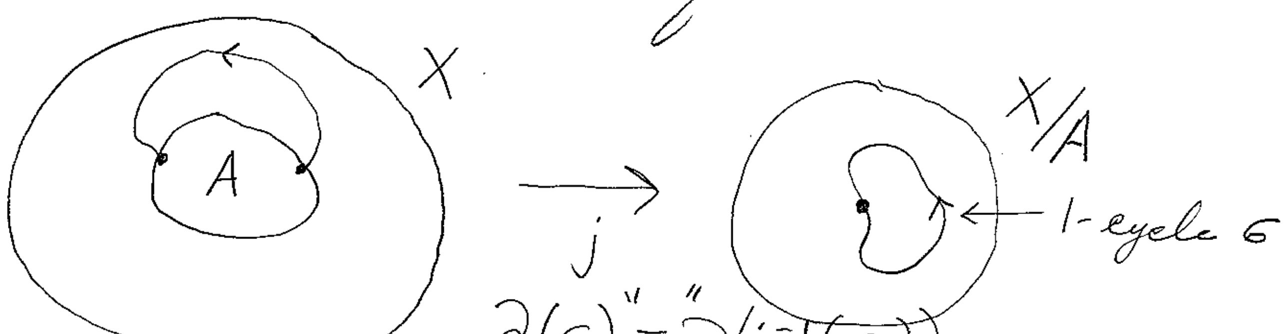
$$\cdots \to \tilde{H}_3(X/A) \xrightarrow{\partial} \tilde{H}_2(A) \xrightarrow{i_*} \tilde{H}_2(X) \xrightarrow{j_*} \tilde{H}_2(X/A) \xrightarrow{\partial} \tilde{H}_1(A)$$
$$\xrightarrow{i_*} \tilde{H}_1(X) \xrightarrow{j_*} \tilde{H}_1(X/A) \xrightarrow{\partial} \tilde{H}_0(A) \xrightarrow{i_*} \tilde{H}_0(X) \xrightarrow{j_*} \tilde{H}_0(X/A) \to 0$$

is <u>exact</u>, where

i_* is induced by the inclusion $i: A \to X$

j_* comes from the projection $j: X \to X/A$

∂ is harder to define.



$\partial(\sigma) " = " \partial(j^{-1}(\sigma))$

$\underline{Ex}$: $X = I$, $A = \partial I$, $X/A = S^1$

$$\to \overset{0}{\tilde{H}_2(I)} \xrightarrow{i_*} \tilde{H}_2(S^1) \xrightarrow{j_*} \overset{0}{\tilde{H}_1(\partial I)} \longrightarrow \overset{0}{\tilde{H}_1(I)} \xrightarrow{j_*} \tilde{H}_1(S^1)$$

$$\xrightarrow{\partial} \underset{\mathbb{Z}}{\tilde{H}_0(\partial I)} \xrightarrow{i_*} \underset{0}{\tilde{H}_0(I)} \to \underset{0}{\tilde{H}_0(S^1)} \to 0$$

Forces $\tilde{H}_2(S^1) = 0$ as $\ker j_* = \tilde{H}_2(S^1)$ and $\operatorname{im} i_* = 0$.

Forces $\tilde{H}_1(S^1) = \mathbb{Z}$ as $\ker i_* = \tilde{H}_0(\partial I)$
$\Rightarrow \partial$ is onto

$$0 \to \tilde{H}_1(S^1) \to \underset{\mathbb{Z}}{\tilde{H}_0(\partial I)} \to 0$$

means ∂ is 1-1 $\Rightarrow \partial$ is an isom.

$\underline{So}$:

$$\tilde{H}_n(S^1) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H_n(S^1) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

<u>Thm</u>: $\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & \text{when } n = k \\ 0 & \text{otherwise} \end{cases}$

(Note true for $n = 0, 1$)

Pf: Induct on n, and use

$$X = D^n, \quad A = \partial D^n = S^{n-1}, \quad X/A = S^n$$

$$\tilde{H}_k = 0 \qquad = \begin{cases} \mathbb{Z} & k = n-1 \\ 0 & \text{otherwise} \end{cases}$$

Only interesting bit

$$\begin{array}{ccccccc} \tilde{H}_n(D^n) & \to & \tilde{H}_n(S^n) & \longrightarrow & \tilde{H}_{n-1}(S^{n-1}) & \to & \tilde{H}_{n-1}(D^n) \\ 0 & & & & \mathbb{Z} & & 0 \end{array}$$

$\Rightarrow \tilde{H}_n(S^n) = \mathbb{Z}$, others are 0. ∎