Lecture: Covering transformations and regular covers.

Recall:

Covering Transformations:

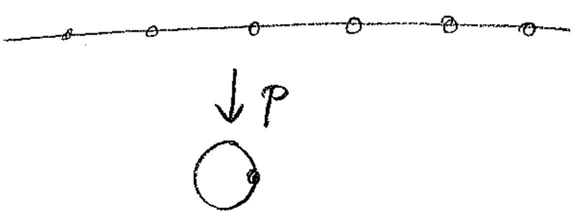
homeomorphism

$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd} \qquad \boxed{p \circ f = p}$$

$G(\tilde{X})$ = group of covering trans

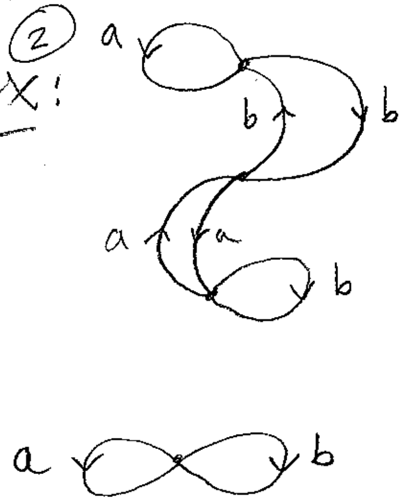
[$\circ p$ is comp of fns. really assoc to p.]

Ex ①:



$G(\tilde{X})$ = trans w/ integer shifts $\cong \mathbb{Z}$

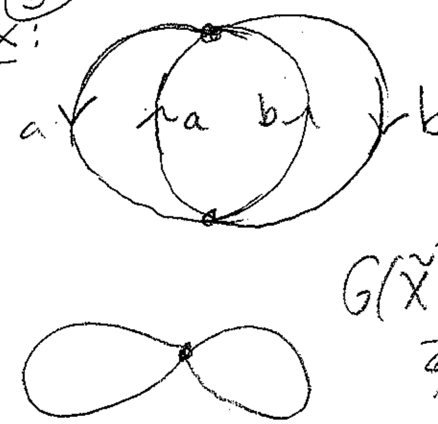
Ex ②:



[Query]

$G(\tilde{X}) = \{id\}$

Ex ③:



$G(\tilde{X}) = \mathbb{Z}/2\mathbb{Z}$

or galios

Def: A connected cover is normal, or regular, if $\forall \tilde{x}_0, \tilde{x}_1 \in \tilde{X}$ with $p(\tilde{x}_0) = p(\tilde{x}_1)$, $\exists f \in G(\tilde{X})$ with $f(\tilde{x}_0) = \tilde{x}_1$. [Ex: ① + ③ but not ②]

Note: For reasonable X, such f exists $\Longleftrightarrow$

$$P_*(\pi_1(\tilde{X}, \tilde{x}_0)) = P_*(\pi_1(\tilde{X}, \tilde{x}_1)) = \gamma \cdot p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \cdot \gamma^{-1}$$

Thm: X path conn, loc. path conn, S.L.S.C.
Then a connected cover $\tilde{X} \xrightarrow{p} X$ is regular
$\Longleftrightarrow p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgp of $\pi_1(X, x_0)$.

Q: What is $\pi_1(X, x_0) / p_*(\pi_1(\tilde{X}, \tilde{x}_0))$?

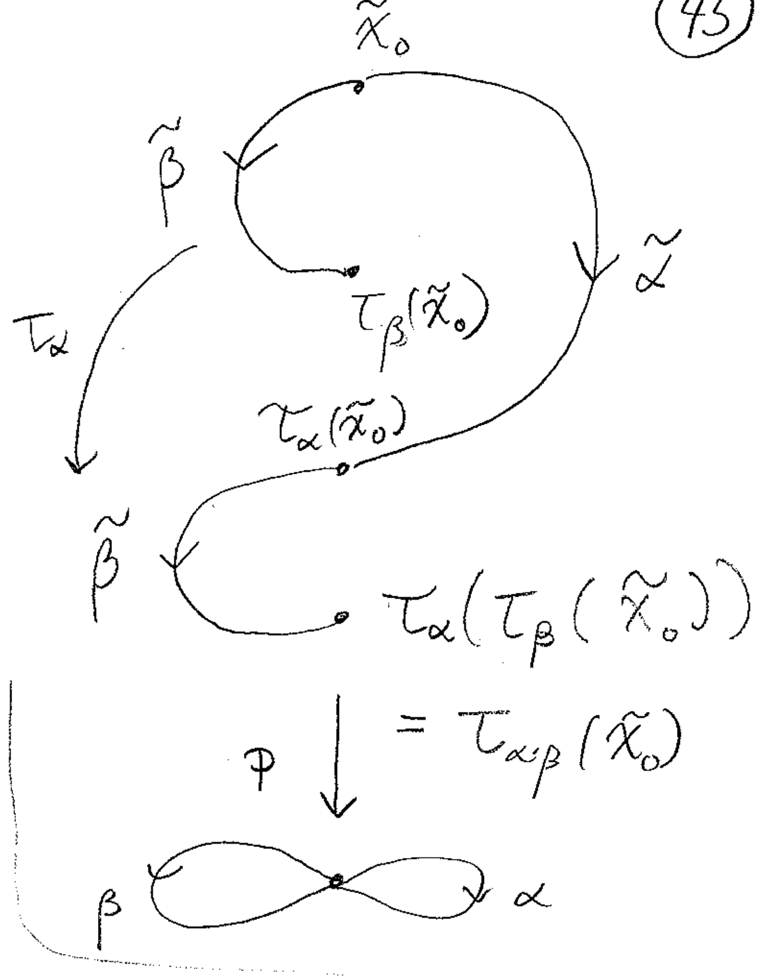
$(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$ a regular cover. Given $\alpha \in \pi_1(X, x_0)$, let τ_α be the unique elt s.t. $\tau_\alpha(\tilde{x}_0) = \tilde{\alpha}(1)$ where $\tilde{\alpha}(0) = \tilde{x}_0$. explain

<u>Note</u>: $\tau_{\alpha \cdot \beta} = \tau_\alpha \circ \tau_\beta$

and

$\tau_\alpha = id \iff$

$\alpha \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$



<u>Thm</u>: $\tilde{X}$ a path conn <u>regular</u> cover. Then

$$\bar{\tau} : \pi_1(X, x_0) \big/ p_*(\pi_1(X, \tilde{x}_0)) \longrightarrow G(\tilde{X})$$

is an isomorphism.

<u>Pf</u>: <u>Onto</u>: $\tilde{X}$ path conn. 1-1: obs. above. ▨

reasonable.

<u>Ex</u>: $\tilde{X}$ a universal cover of X, get

$\pi_1(X, x_0) = G(\tilde{X})$ as $\langle 1 \rangle$ is normal.

Moreover, $X = \tilde{X} \big/ G(\tilde{X}) = \pi_1 X$.

Ex: $S^1 = \mathbb{R}/\mathbb{Z}$,  $= \mathbb{R}^2/\mathbb{Z}^2$, 

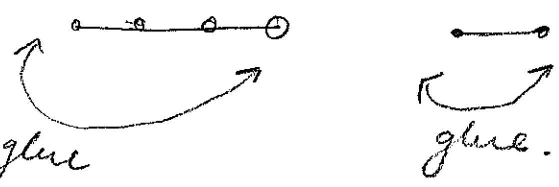
$\mathbb{R}P^2 \vee \mathbb{R}P^2 =$  $/ D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$

etc...

Also: If $H \leq \pi_1(X, x_0)$ then the cover cor. to H is $\tilde{X}/\tau(H)$.

Ex: $S^1 \xrightarrow{p} S^n$ comes from $\mathbb{R}/3\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$

$z \longmapsto z^3$



Important note:

$\tau : \pi_1(X, x_0) \longrightarrow G(\tilde{X})$ gives an action of π_1 on $p^{-1}(x_0)$. This has nothing to do the lifting correspondence from Monday.