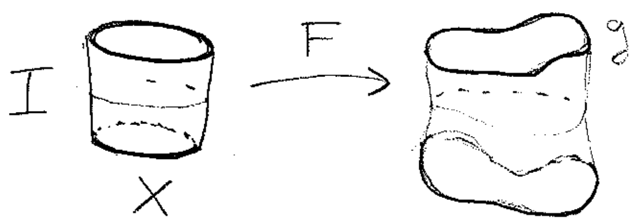
Lecture 8: Deforming spaces.

<u>Last time</u>: $h:(X,x_0)\longrightarrow(Y,y_0)$ gives a <u>homomorphism</u> $h_*:\pi_1(X,x_0)\longrightarrow\pi_1(Y,y_0)$

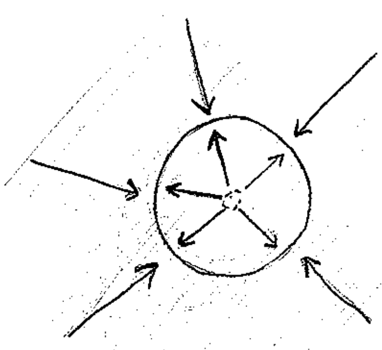
$$[f]\longrightarrow[h\circ f]$$

<u>Previously</u>: $f,g:X\longrightarrow Y$ are homotopic if $\exists\, F:X\times I\longrightarrow Y$ with $F(x,0)=f(x)$ and $F(x,1)=g(x)$ for all $x\in X$



<u>Def</u>: A <u>retraction</u> of a space X to a subspace A is a map $r:X\to A$ with $r|_A=id_A$.

<u>Ex</u>: $X=\mathbb{R}^2\setminus\{0\}$, $A=S^1$, $r(x)=\dfrac{x}{|x|}$

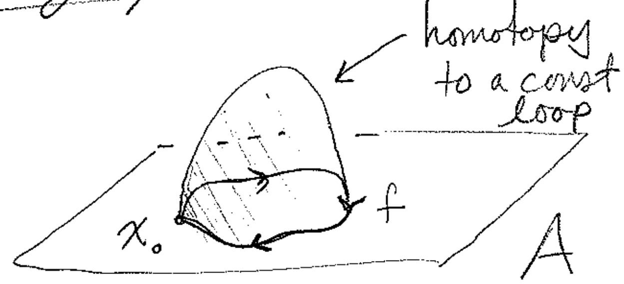


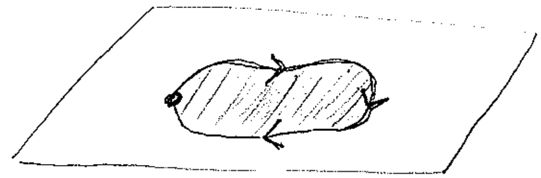
<u>Lemma</u>: If X retracts to A, then $\pi_1(A,x_0)\xrightarrow[i_*]{}\pi_1(X,x_0)$ is 1-1 where $i:A\hookrightarrow X$ is inclusion.

<u>So</u>: $\pi_1 X\supseteq\mathbb{Z}$. [In fact, they're equal...]

Pf by picture:

Use r to push homotopy down into A





$[f]$ in $\ker(i_*)$ $\Longrightarrow$ $[f] = 1$.

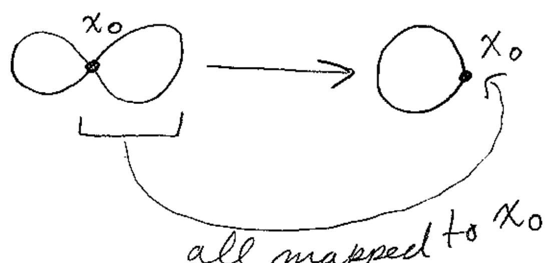
Pf by Algebra: Have

$$\pi_1(A, x_0) \xrightarrow{i_*} \pi_1(X, x_0) \xrightarrow{r_*} \pi_1(A, x_0)$$

$$r_* \circ i_* = (r \circ i)_* = (id_A)_* = id$$

and so i_* is 1-1. $\blacksquare$

Note: When X retracts to A, don't always have $\pi_1 A \cong \pi_1 X$. E.g. r :



Notation: For

$$F : X \times I \longrightarrow Y$$

there is an assoc. *family of maps* $f_t : X \to Y$ defined by $f_t(x) = F(x,t)$. [E.g. F is a homotopy between f_0 and f_1]

Def: X *deformation retracts* to A, if there is a family of maps $f_t : X \to X$ with $f_0 = id_X$, $f_1(X) \subseteq A$, and $f_t|_A = id_A$ for all t.

Ex:

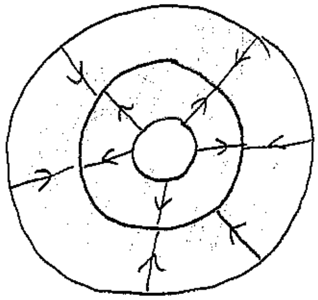
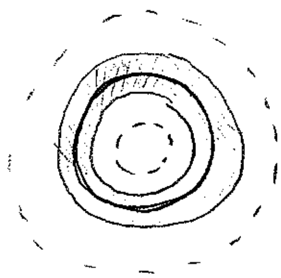


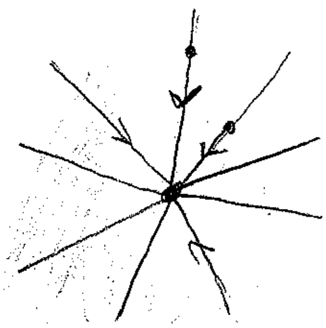
Image of $f_{1/2}$



$X = \{ 1/2 < |x| < 3/2 \}$

$A = S^1$

Ex: $X = \mathbb{R}^2$, $A = \{\vec{0}\}$ where $f_t(x) = (1-t)x$

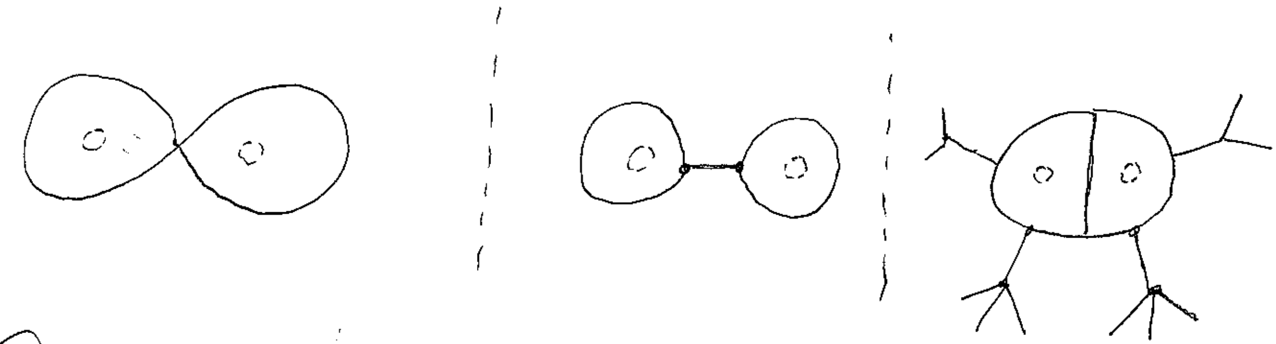


Ex: $X = \mathbb{R}^2 \setminus \{0\}$, $A = S^1$ (HW!)

Thm: If X def. retracts to A, then $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is an isomorphism, where $x_0 \in A$.

Proof: If $[g] \in \pi_1(X, x_0)$, then $g = f_0 \circ g \simeq_p f_1 \circ g$ since $f_t(x_0) = x_0$ for all t. So $i_*([f_1 \circ g]) = [g]$. So i_* is onto. ∎

Ex: The following subspaces are def. retracts of $X = \mathbb{R}^2 \setminus \{\text{two pts}\} \Longrightarrow$ all have $\pi_1 = F_2$ ← free gp on two gens.



Def: X and Y are homotopy equivalent if $\exists$ maps $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ with $g \circ f \simeq id_X$ and $f \circ g \simeq id_X$. Write $X \simeq Y$.

Ex: Suppose X def. retracts to A via f_t. Then $X \simeq A$, via $X \underset{i}{\overset{f_1}{\rightleftarrows}} A$ since $f_1 \circ i = id_A$ and $i \circ f_1 = f_1 \simeq f_0 = id_X$.

Note: homotopy equivalence is an equivalence relation. Homotopy type is what Alg. Top "sees".

Fact: $X \simeq Y$ iff $\exists$ a space $Z \supset A, B$ with $X \cong A$, $Y \cong B$ and Z def. retracting to both A and B.

<u>Thm</u>: If $f: X \to Y$ is a homotopy equivalence, then $f_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$ is an isomorphism.

<u>Idea</u> (ignoring base points!)

<u>Lemma</u>: If $f_t: X \to Y$ is a family of maps, then $(f_0)_* = (f_1)_*: \pi_1 X \to \pi_1 Y$.

Then

$$\pi_1 X \underset{g_*}{\overset{f_*}{\rightleftarrows}} \pi_1 Y$$

and $g_* \circ f_* = (g \circ f)_* = (id_X)_* = id_{\pi_1 X}$.

Similary, $f_* \circ g_* = id_{\pi_1 Y}$. Thus f_* and g_* are inverse bijections.

For details, see Prop 1.18 in Hatcher.