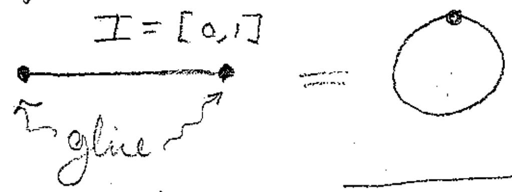
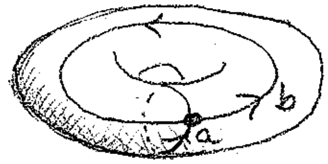
Lecture 9: Quotient topology + cell complexes.

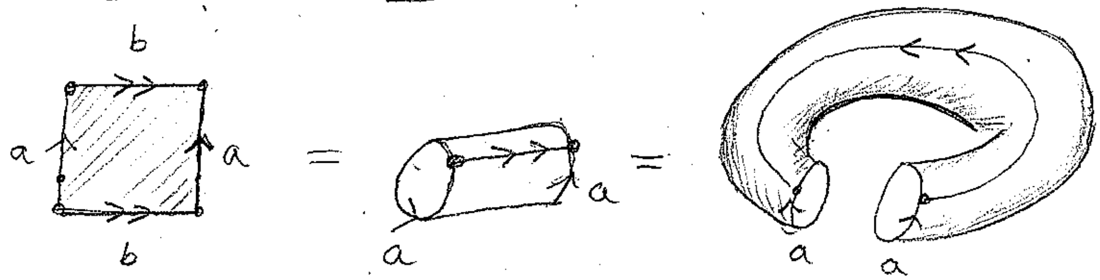
Identification Spaces:

Ex: 



$= S^1 \times S^1$

$\|$

Ex: 

[What does "gluing" or "idenifying" mean abstractly?]

X a top. space with an equivalence relation $\sim$.

Ex: $X = I$: $0 \sim 1$ all other pts equiv to only themselves.

Ex: $X = I \times I$: $(0,t) \sim (1,t)$ and $(s,0) \sim (s,1)$

[forces $(0,0) \sim (1,0) \sim (0,1) \sim (1,1)$]

Consider $Y = X/\sim$ = set of equiv. classes

and the map $p: X \to Y$.

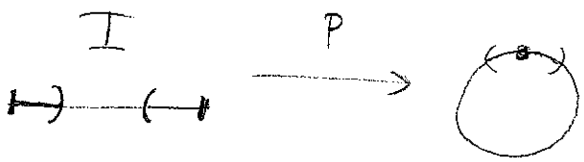
The quotient topology on Y has open sets

$$\mathcal{U} = \{ U \subseteq Y \mid p^{-1}(U) \text{ is open in } X \}$$

Easy Check: This is a topology on $Y = X/\sim$

Note: $p: X \to Y$ is continuous, [and the quotient top. has the most open sets among all top for which this is true.]

Ex: $I \xrightarrow{p}$



Thus $I/\sim \,\cong S^1$

Ex: $f: \tilde{X} \longrightarrow X$ an onto covering map. On $\tilde{X}$, set $\tilde{x} \sim \tilde{x}'$ iff $f(\tilde{x}) = f(\tilde{x}')$. Then we have a bijection $\tilde{X}/\sim \;\Longleftrightarrow X$ and this is a homeomorphism.

Basic Fact Here $Y = X/\sim$ with the quotient top.

① $f: Y \to Z$ is continuous $\Longleftrightarrow$ $f \circ p: X \to Z$ is.

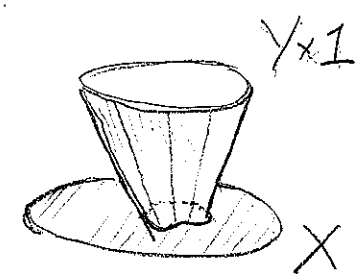
$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "f\circ p"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

Ex: $\mathbb{RP}^n = \{x \in \mathbb{R}^{n+1} \setminus \{0\}\} \big/ \begin{matrix} x \sim \lambda x \\ \text{for } \lambda \in \mathbb{R}^+ \end{matrix}$

Ex: $X \vee Y = X \amalg Y \big/_{x_0 \sim y_0}$

Ex: Mapping Cylinder: $f: X \to Y$ some map.

Then $C_f = (X \times I) \amalg Y \big/_{(x,0) \sim f(x)}$

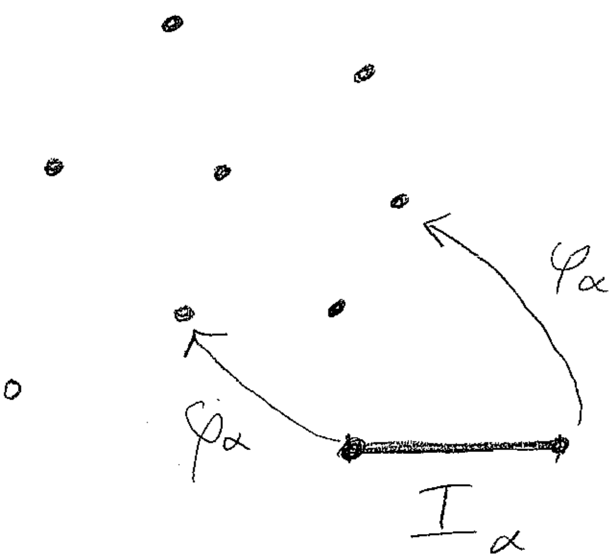


Note: C_f deformation retracts to X.

Fact: If f is a homotopy equiv, then C_f also def retracts to $Y \times \{1\}$.

Graph: [as top spaces.]

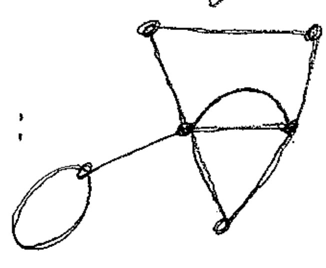
vertices: X^0 a discrete set of points

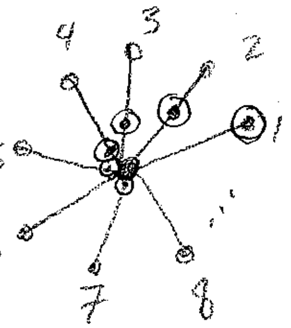


edges: $\coprod_\alpha I_\alpha$, $\varphi_\alpha : \partial I_\alpha \to X^0$

$$X = X^0 \coprod \left(\coprod_\alpha I_\alpha \right) \Big/ \; x \sim \varphi_\alpha(x) \text{ for } x \in \partial I_\alpha$$

with the quotient top.

Ex: 

Ex: 

Q: Does the sequence $x_n = 1/n$ on n^{th} edge converge to the center?

A. No! $\Rightarrow$ this topology *doesn't* come from a metric.

CW Complex: X^n — built from cells $D^k = \{x \in \mathbb{R}^k \mid |x| \le 1\}$ of dim $\le n$

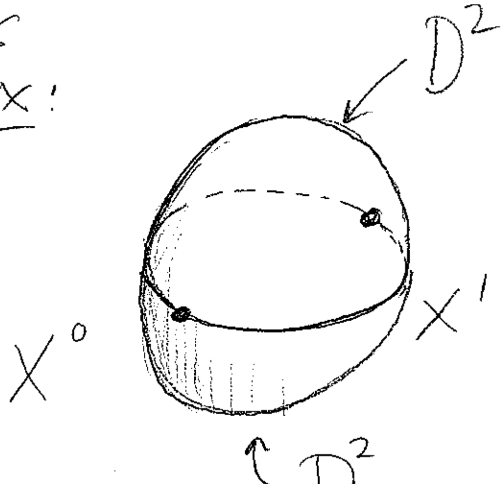
$$X^{n+1} = X^n \amalg \left(\coprod_\alpha D_\alpha^{n+1} \right) \Big/ {\scriptstyle x \in \partial D_\alpha^{n+1} \sim \varphi_\alpha(x).}$$

$\varphi_\alpha : \partial D_\alpha^{n+1} \to X^n$

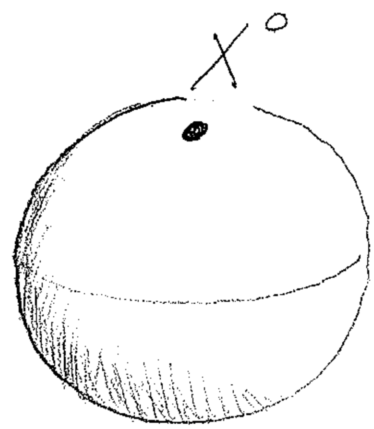
"attaching maps"

$$\boxed{X = \bigcup X_n}$$

Ex:



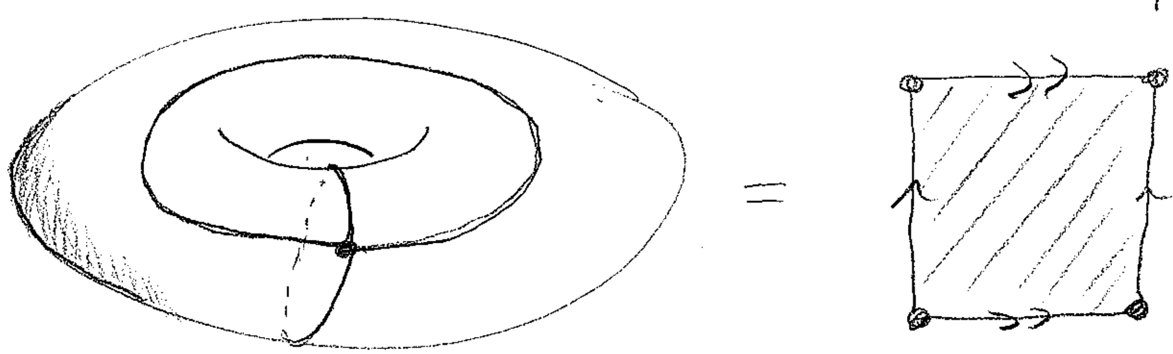
Ex:



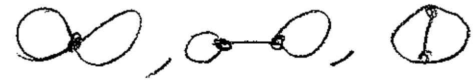
$X^1 = X^0, \quad X^2 = X^0 \amalg D^2$

with $\varphi : \partial D^2 \to pt$

Ex:



Note: Can have cells in arb. large dims.

Last time, saw that  are all $\simeq$, and so have $\pi_1 = F_2$. Next time:

Thm π_1(Graph) is a free group.

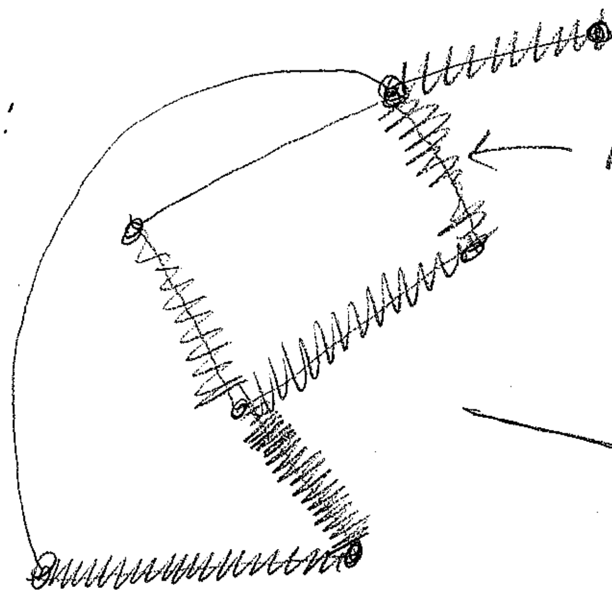
Once we've mastered covering spaces, this shows that

Cor: Any subgp of a free gp is also free.

Thm follows from

Fact: Any connected graph is $\simeq$ to $\bigvee_\alpha S^1$ ← π_1 = Free Group on the index set of α's.

Idea:

$X =$  ← A = max. tree

$\longrightarrow X/A = X/\sim$ where $x \sim x'$ if both are in A.

should be a hom. equivalence.

Need: Crit for when $X \to X/A$ is a $\simeq$.