Lecture 5:

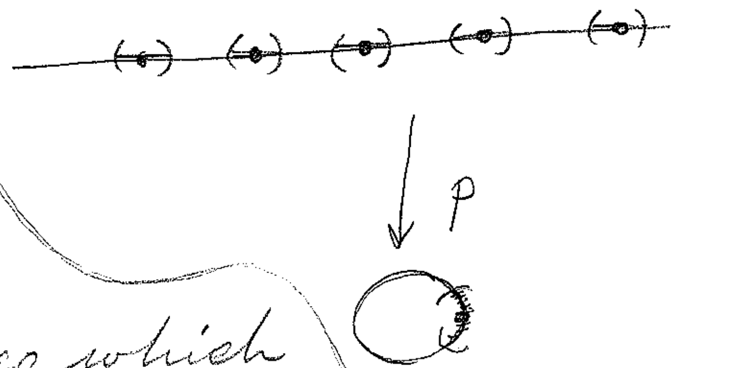


Last time: $p: \tilde{X} \to X$ evenly covers $U \overset{\text{open}}{\subseteq} X$ if $p^{-1}(U) = \coprod_\alpha V_\alpha$ with $V_\alpha \overset{\text{open}}{\subseteq} \tilde{X}$ and each $p|_{V_\alpha}$ is a homeo.

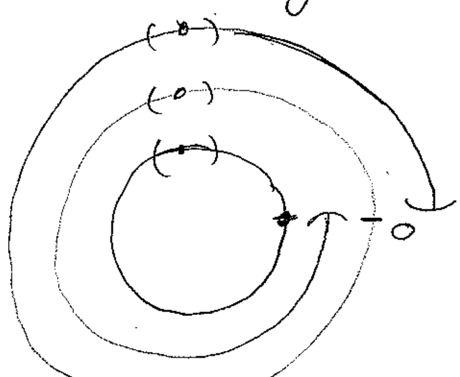
disjoint union

$p: \tilde{X} \to X$ is a covering map if every $x \in X$ has an evenly covered nbhd.

Ex: $p: \mathbb{R} \to S^1$ $p(t) = e^{-2\pi t i} = (\cos 2\pi t, -\sin 2\pi t)$



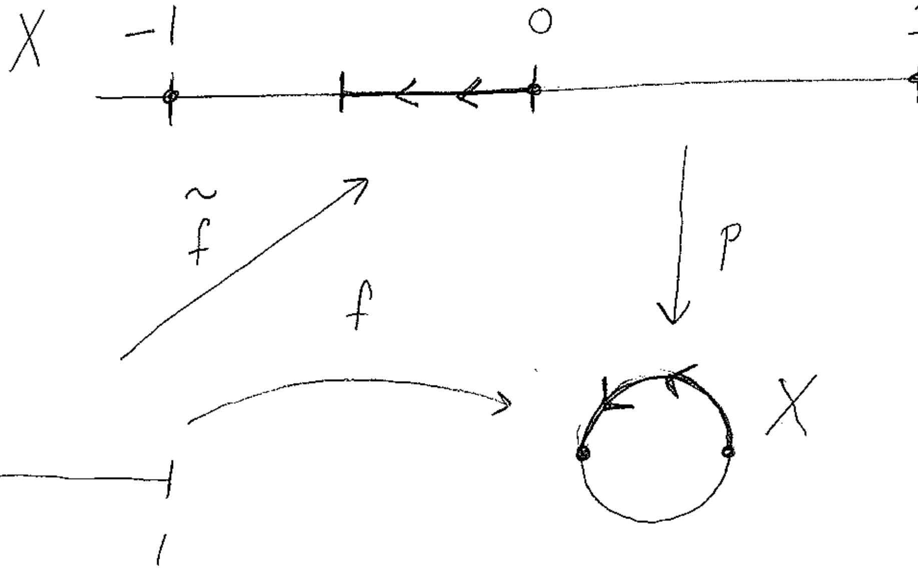
A covering map is a local homeo, but $p|_{(-1,1)}$ is a local homeo which is not a covering map — 1 is not evenly covered.



Def: Let $p: \tilde{X} \to X$ and $f: Y \to X$ be maps. A lift of $f$ is a map $\tilde{f}: Y \to \tilde{X}$ s.t. $p \circ \tilde{f} = f$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

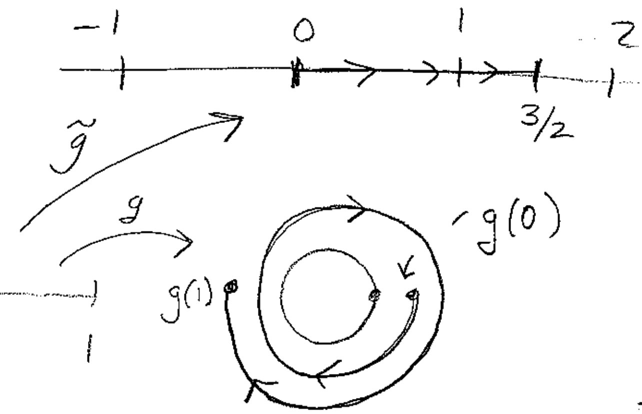
**Ex:**




$f(s) = e^{\pi s i}$

$\tilde{f}(s) = -\frac{s}{2}$

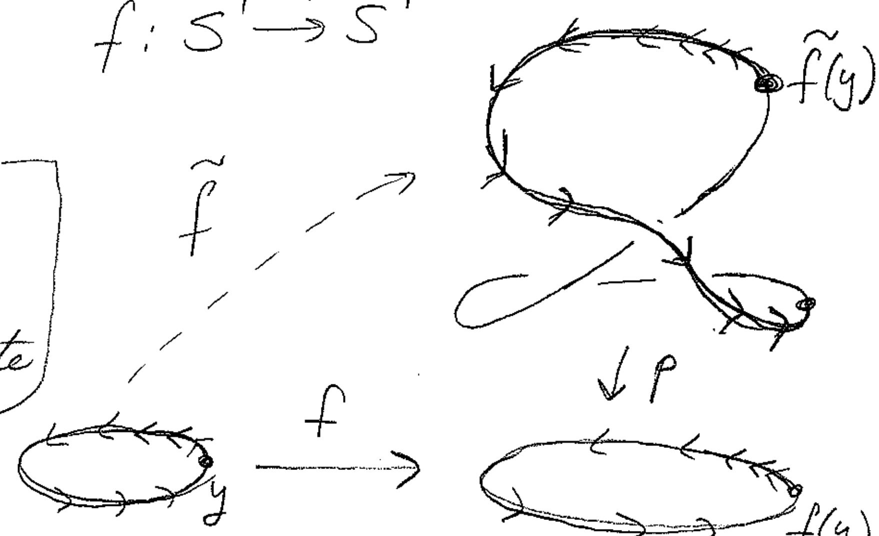
**Ex:**




N.B. you can see how we're going to use lifts to formalize winding number

**Non Ex:** $p: S^1 \longrightarrow S^1$
$z \longmapsto z^2$

$f: S^1 \xrightarrow{id} S^1$




[The fund gps will tell use exactly when lifts exist, can use to compute $\pi_1$. / Note non-uniqueness of lifts in 1st example]

No lift exists

**Thm:**

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p \text{ covering map}"] \\ I \arrow[r, "f\text{-path starting at } x_0"'] \arrow[ur, dashed, "\tilde{f}"] & X \end{tikzcd}$$

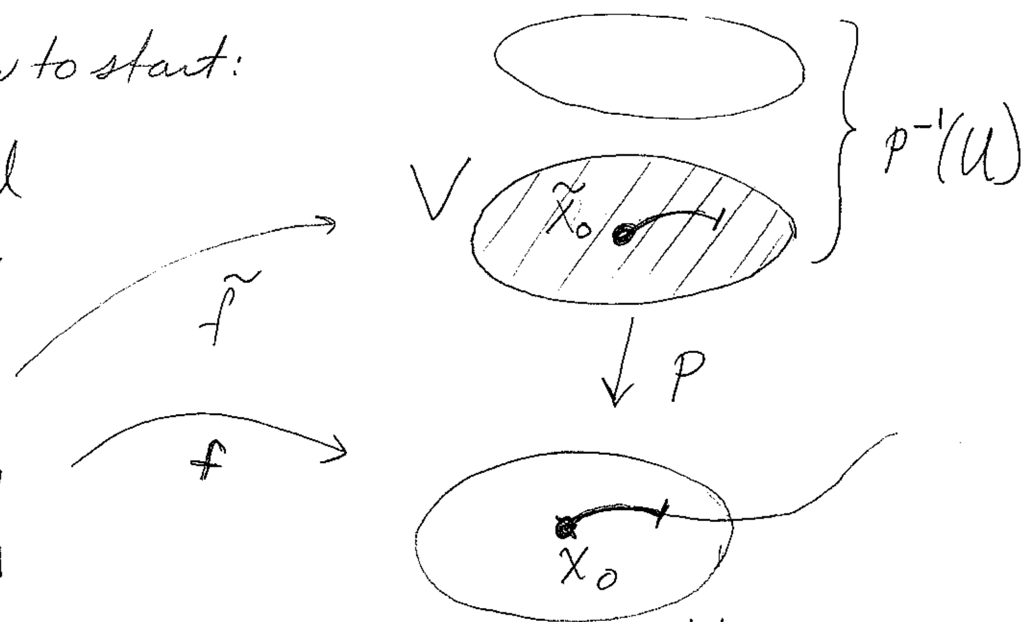
For each $\tilde{x}_0 \in p^{-1}(x_0)$, $\exists$ a unique lift of $f$ to a path starting at $\tilde{x}_0$.

*Pf:* Existence: How to start:

$f([0,s_1]) \subseteq U$

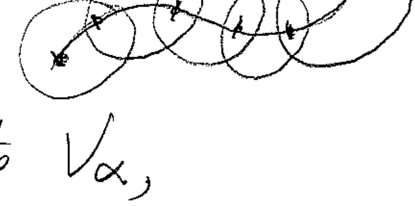
Define $\tilde{f}$ on $[0,s_1]$

by $\tilde{f} = (p|_V)^{-1} \circ f$


U evenly covered

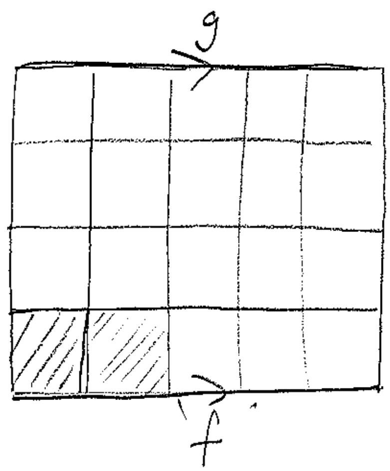
Now repeat using that compactness gives a partition $0 = s_0 < s_1 < s_2 < \cdots < s_n = 1$ where each $f([s_k, s_{k+1}]) \subseteq U_k$ an evenly covered open set.

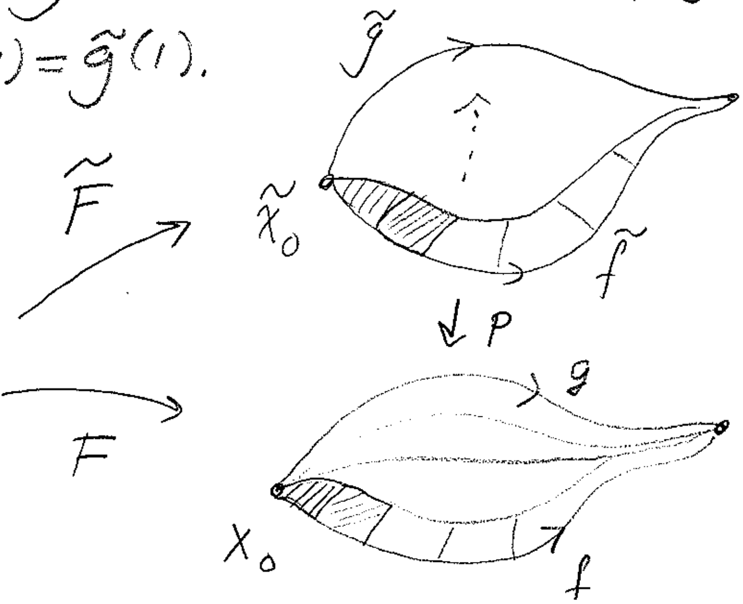


Uniqueness: Given that $p^{-1}(U)$ is the disjoint union of sets $V_\alpha$, $\tilde{f}([0,s_1])$ must lie in only one of them, the one containing $\tilde{x}_0$. Since $p|_V$ is 1-1, this makes $\tilde{f}$ unique. ∎

*Addendum*: Suppose $f \simeq_p g$, and $\tilde{g}$ is the lift of $g$ starting at $\tilde{x}_0$. Then $\tilde{f} \simeq_p \tilde{g}$. In particular, $\tilde{f}(1) = \tilde{g}(1)$.

*Pf*:




*Prop 1.30*: $p: \tilde{X} \to X$ a covering map, $f: Y \to X$ a map, $F: Y \times I \to X$ a homotopy of $F$. If $f$ lifts to $\tilde{f}$ then $\exists$ a unique lift of $\tilde{F}$ to a homotopy of $\tilde{f}$.

[*Note*: Thm was $Y = pt$, Addendum was $Y = I$]

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Let $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map

*Def*: The lifting *correspondence*

$\phi: \pi_1(X, x_0) \to p^{-1}(x_0)$ is

$\phi([f]) =$ end pt of the lift $\tilde{f}$ of $f$ based at $\tilde{x}_0$.

Def: $Z$ is simply connected if it is path connected and $\pi_1 Z = 1$.

[Ex: $\mathbb{R}^n$]

Thm: $p: \tilde{X} \to X$ is a covering map w/ $\tilde{X}$ simply connected. Then $\phi$ is bijection.

Pf: Next time.

Thm: $\pi_1(S^1, 1) = \mathbb{Z}$

Pf: Consider the cover $p: \mathbb{R} \to S^1$

By the thm, we have a bijection

$$\pi_1(S^1, 1) \xrightarrow{\phi} p^{-1}(1) = \mathbb{Z}$$

Claim: $\phi$ is a homomorphism.

Note $\phi([f^n]) = n$, so as $\phi$ is a bijection, we see that $\pi_1(S^1)$ is cyclic, generated by $[f]$

f