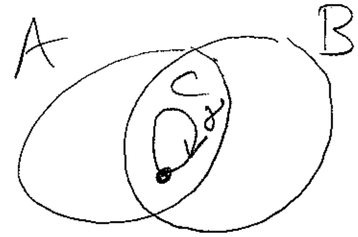
Lecture 12: Van Kampen's Thm.



Today:

Thm: $X = A \cup B$ where A, B and $C = A \cap B$ are path connected open sets. For $c_0 \in C$, have

$$\pi_1(X, c_0) = \frac{\pi_1(A, c_0) * \pi_1(B, c_0)}{\langle i_{A_*}(\gamma)\, i_{B_*}(\gamma)^{-1} \text{ for } \gamma \in \pi_1 C \rangle}$$

where i_A, i_B are the inclusions.

Def: G_1, G_2 groups. Then their free product is

$$G_1 * G_2 = \{ \underbrace{g_1 * g_2 * \cdots * g_n}_{\text{reduced word.}} \mid g_i \in G_1 \text{ or } G_2,\ g_i \text{ and } g_{i+1} \text{ not in the same group} \}$$

Op is concatenation + reduction

Ex: $G_i \cong \mathbb{Z}$, say $G_1 = \langle a \rangle$ and $G_2 = \langle b \rangle$.
Then $G_1 * G_2 = \text{FreeGroup}(a, b)$

Ex: $G_1 = \langle a \mid a^2 = 1 \rangle \quad G_2 = \langle b \mid b^3 = 1 \rangle$

Typical mult works like this

$$(a * b * a * b^{-1}) * (b * a * b) =$$

$$a * b * a * \underbrace{b * b^{-1}}_{= bb^{-1} = 1} * a * b = a * b * a * a\, b$$

$$= a * b * b = a * b^{-1}.$$

<u>Universal Characterization:</u>

$F = G_1 * G_2$ is the group with

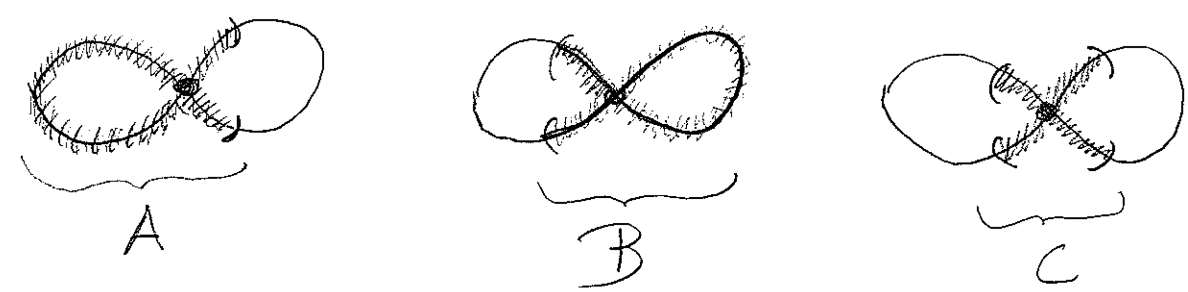
$\rho_i : G_i \hookrightarrow F$ s.t. $\forall$ pairs $f_i : G_i \to S$ ← some gp

$\exists!\, f : F \to S$ completing

$$\begin{tikzcd} F & G_2 \arrow[l] \arrow[d, "f_2"] \\ G_1 \arrow[u] \arrow[r, "f_1"'] & S \arrow[from=1-1, dashed, "f"] \end{tikzcd}$$

<u>Ex's of Van Kampen</u>

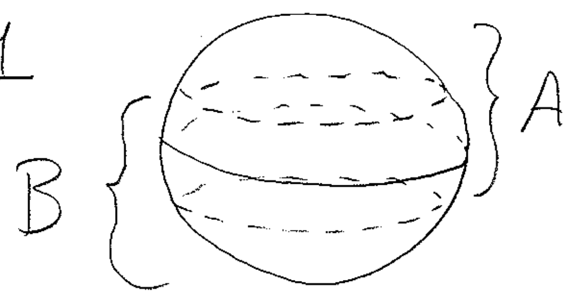
① $X =$



$\pi_1 A \cong \pi_1 B \cong \mathbb{Z} \quad \pi_1 C = 1$

$\pi_1 X = \pi_1 A * \pi_1 B = \mathbb{Z} * \mathbb{Z} = F_2.$

② $\pi_1 S^2 = 1$

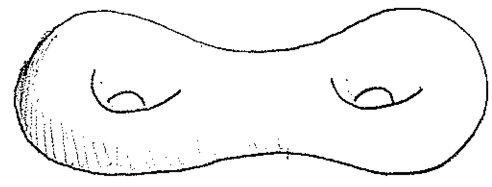


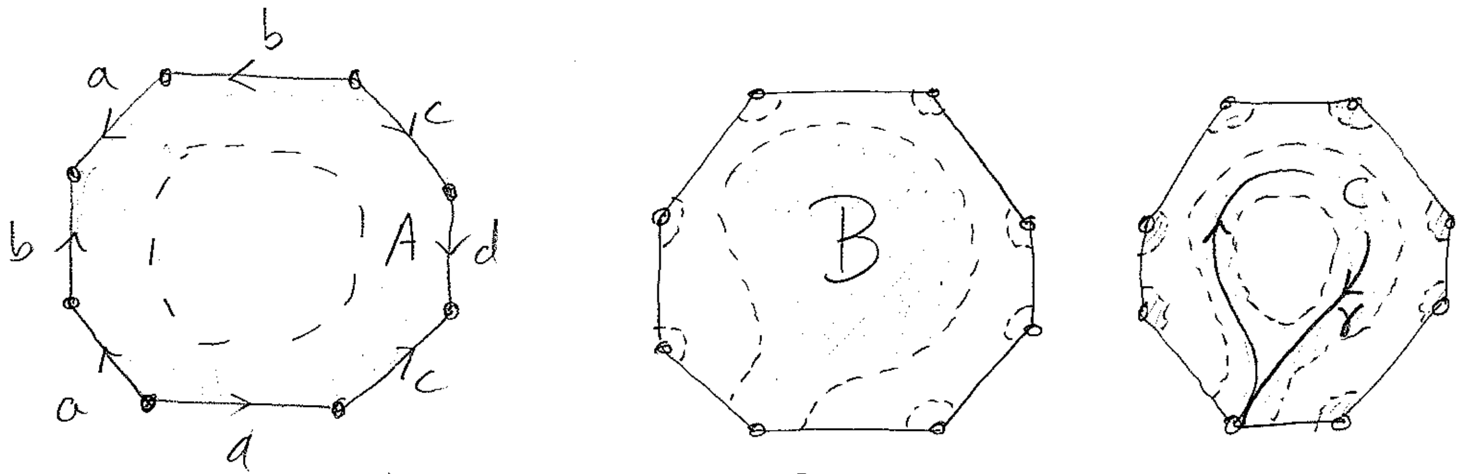
$\pi_1 S^2 = \pi_1 A * \pi_1 B / \langle\ \rangle$

$= 1 * 1 / \langle\ \rangle$

$= 1.$

Note alt. proof.

Ex: $X =$ 



$\pi_1 A \cong \pi_1 X' =$ Free Gp(a,b,c,d)

$\pi_1 B = 1$

$\pi_1 C \cong \mathbb{Z} = \langle \gamma \rangle$

$$\pi_1 X = \pi_1 A * \pi_1 B \Big/ \langle i_A(\gamma)\, i_B(\gamma)^{-1} \rangle$$

$$= \langle a,b,c,d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle$$

Note: No algorithm to decide if a finitely pres. gp is trivial

Trick: Abelianize $\quad G^{ab} = G \big/ [G,G]$

↑ subgp gen by $[g,h] = ghg^{-1}h^{-1}$ for $g,h \in G$

$$\pi_1 X^{ab} = \mathbb{Z}^4$$

So  $\not\cong$  $\not\cong$ .

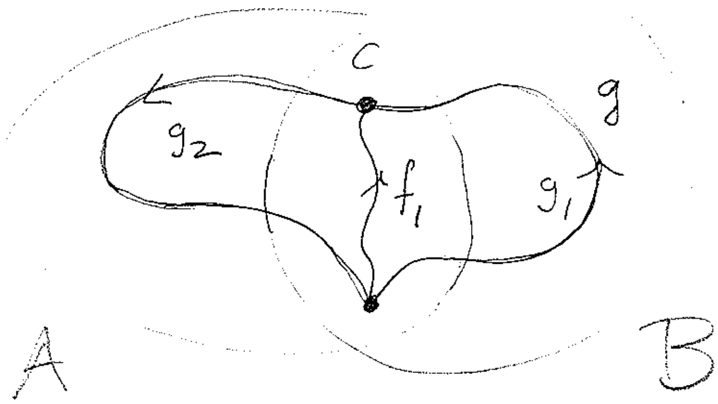
Some ideas behind Van Kampen's Thm: (see Hatcher for details.)

$$\pi_1 A * \pi_1 B \longrightarrow \pi_1 X$$

$$[g_1] * [g_2] * \cdots * [g_n] \longmapsto [g_1] \circ [g_2] \circ \cdots \circ [g_n]$$

↑ formal prod ↑ usual concat. op.

is onto because of



$g \simeq_p g_1 \cdot g_2$

$\simeq_p g_1 \circ \bar{f}_1 \circ f_1 \circ g_2$

$= [g_1 \circ \bar{f}_1] \cdot [f_1 \circ g_2]$

$\quad\ \ \cap \qquad\qquad \cap$

$\quad \pi_1 B \qquad\quad \pi_1 A$

Clearly, the kernel of this map contains $i_A(\gamma) * i_B(\gamma)^{-1}$, and can show they normally generate. $\blacksquare$

X a gen CW cplx. Then

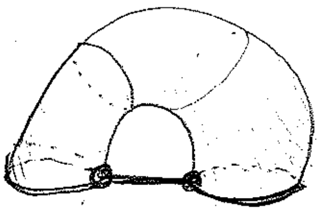
$\pi_1 X' \xrightarrow[i_*]{} \pi_1 X$ is onto,

and $\pi_1 X^2 \xrightarrow[i_*]{} \pi_1 X$ is an $\cong$.

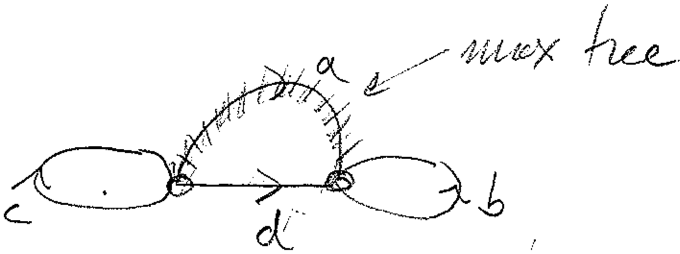
Basically the same reason as last time.

Gen method for computing π_1 of a CW complex.

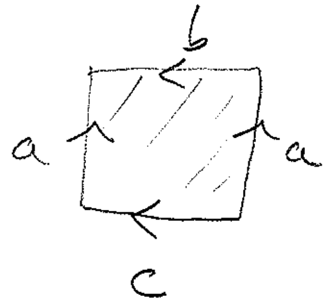
- Look at X^2 only
- Collaspe a max tree in X^1 to a pt.
- Read off relations from 2-cells.



$X^1 =$

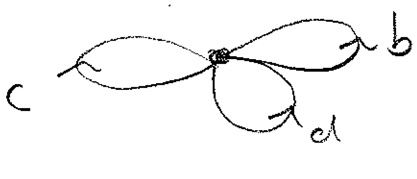


X^2 adds



Crush max tree:

X^2



$\cup$

$$\Rightarrow \pi_1 = \langle b, c, d \mid bc = 1 \rangle = \langle b, d \rangle$$