Lecture 13: Back to covering spaces

Last time:

Van Kampen: $X = A \cup B$ w/ A, B & $C = A \cap B$ path connected open sets. Then

$$\pi_1 X = \pi_1 A * \pi_1 B \Big/ \left\langle i_{A_*}(\gamma)\, i_{B_*}(\gamma)^{-1} \text{ for } \gamma \in \pi_1 C \right\rangle$$

Application:

X = CW complex

Prop: $\pi_1 X^1 \longrightarrow \pi_1 X$ is onto

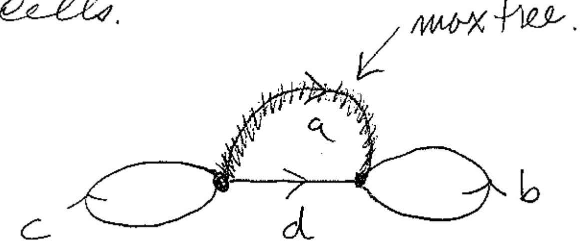
$\pi_1 X^2 \longrightarrow \pi_1 X$ is an $\cong$.

Idea: Can push loops / homotopies off higher dimensional cells.

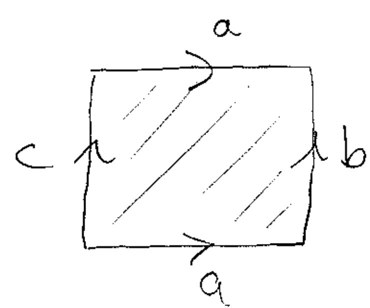
Ex:

X

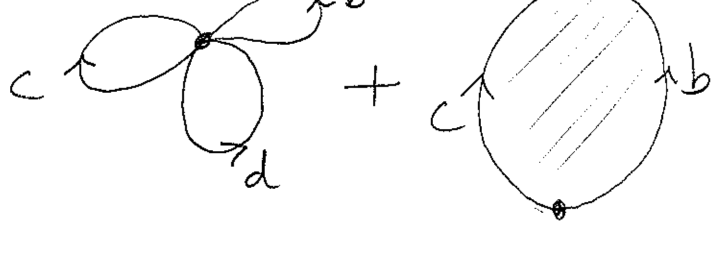
$X^1 =$



X^2 adds



Crush max tree:



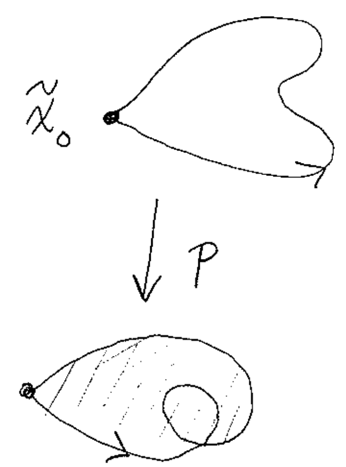
$$\Rightarrow \pi_1 X = \langle b, c, d \mid c b^{-1} = 1 \rangle = \text{Free Gp}(b, d).$$

[Outline remaining topics about covering spaces, esp subgps $\longleftrightarrow$ covers, use of π_1 to solve the lifting problem.]

Prop: $p:(\tilde{X},\tilde{x}_0)\longrightarrow(X,x_0)$ covering map.
Then $p_*:\pi_1(\tilde{X},\tilde{x}_0)\longrightarrow\pi_1(X,x_0)$ is 1-1.

[Hence can ident $\pi_1(\text{cover})$ as a subgp of $\pi_1(X)$.]

Pf: Suppose $[\tilde{f}]\in\ker(p_*)$.
Then $p\circ\tilde{f}\simeq_p \text{const}_{x_0}$. Since $\text{const}_{\tilde{x}_0}$ is a lift of const_{x_0}, the lifting lemma says $\tilde{f}\simeq\text{const}_{\tilde{x}_0}$, i.e. $[\tilde{f}]=1$ in $\pi_1(\tilde{X},\tilde{x}_0)$. ▨



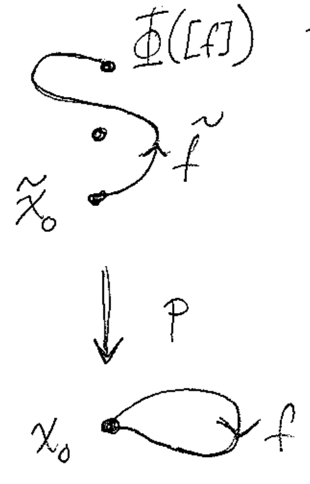
$$[\tilde{f}] \overset{p_*}{\longmapsto} [p\circ\tilde{f}]$$

Recall: lifting correspondence $\Phi:\pi_1(X,x_0)\longrightarrow p^{-1}(x_0)$
$[f]\longrightarrow$ end of the lift of f starting at $\tilde{x}_0$

Note: $p_*(\pi_1(\tilde{X},\tilde{x}_0))$

$=$ loops at x_0 which lift to loops at $\tilde{x}_0$

$=\Phi^{-1}(\tilde{x}_0)$



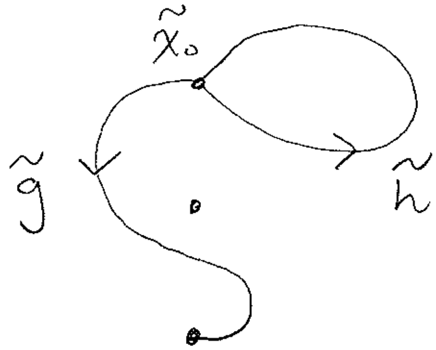
Thm: $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$ a cover with $\tilde{X}$ and X path connected.

Then

$$\pi_1(X, x_0) \Big/ p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \xrightarrow{\ \Phi\ } p^{-1}(x_0) \text{ is a bijection.}$$

Ex: $p: S^1 \to S^1$, $z \mapsto z^n$ $\quad p_*(\pi_1 S^1) = n\mathbb{Z}$ which has index n

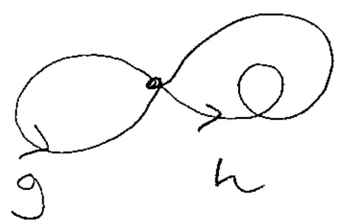
$p^{-1}(1) = n$ pts.



Pf: Let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

As $\Phi(hg) = \Phi(g)$ for $g \in H$,

Φ makes sense as a function of right cosets.



Onto: Since $\tilde{X}$ is path connected.

1-1: Suppose $\Phi(f) = \Phi(g)$, then

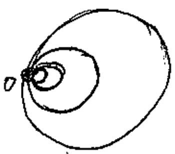
fg^{-1} lifts to a loop at $\tilde{x}_0$.

$\Rightarrow fg^{-1} \in H \Rightarrow Hfg^{-1} = H \Rightarrow Hf = Hg$. ▨

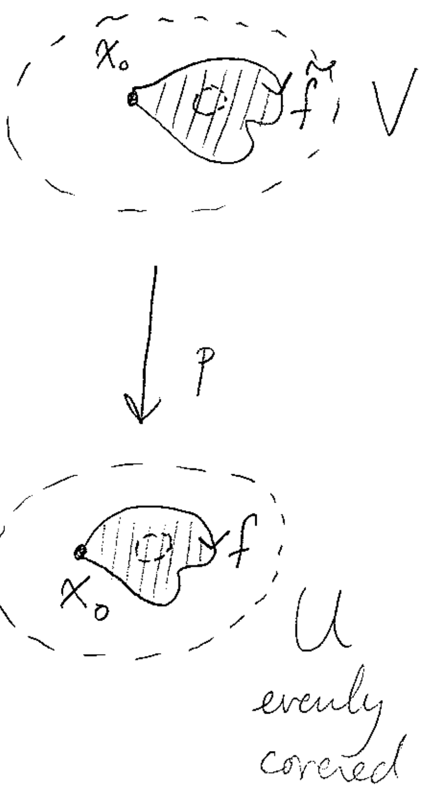
Goal: For X "reasonable", given $H \leq \pi_1 X$,
$\exists!$ cover with $p_*(\pi_1 \tilde{X}) = H$. (cf. $X = S^1$).

Q: What about $H = 1$? E.g. when does X have a simply conn. cover? (called the "universal cover")

Yes: , , ,  [*Q*: What is the cover?]

No: x_0  $= H$ What about  ?

Suppose $\tilde{X}$ is a simp. conn. cover of X.



Then $i_*(\pi_1(U, x_0)) = \langle 1 \rangle$ in $\pi_1(X, x_0)$.

For $H = x_0$ , no open nbhd of x_0 has this prop, so doesn't have a cover with $\pi_1 \tilde{X} = 1$.

Def: X is <u>semi-locally simply connected</u> if every $x \in X$ has a nbhd U s.t.

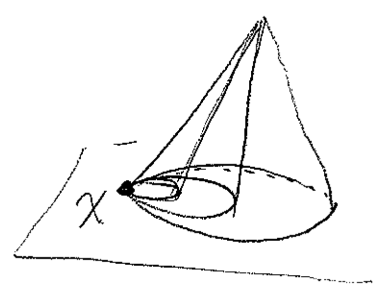
$$i_*(\pi_1(U,x)) = 1 \text{ in } \pi_1(X,x).$$

<u>Locally simply connected</u>: Given a nbhd V of x, $\exists$ a nbhd $U \subseteq V$ with $\pi_1(U,x) = 1$.

Ex: 

is L.S.C.

Ex: S.L.S.C but not L.S.C.: Cone(H)



Thm: X path conn, loc. path conn, S.L.S.C. Then X has a universal cover.

Cor: So  has a univ. cover.

Q: What is it?