Lecture 6: Computing π_1 via the lifting correspondence

Last time:

Thm:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p \text{ covering map}"] \\ I \arrow[ur, "\tilde{f}"] \arrow[r, "f\text{-path starting at } x_0"'] & X \end{tikzcd}$$

For each $\tilde{x}_0 \in p^{-1}(x_0)$, $\exists$ a unique lift of f to a path starting at $\tilde{x}_0$

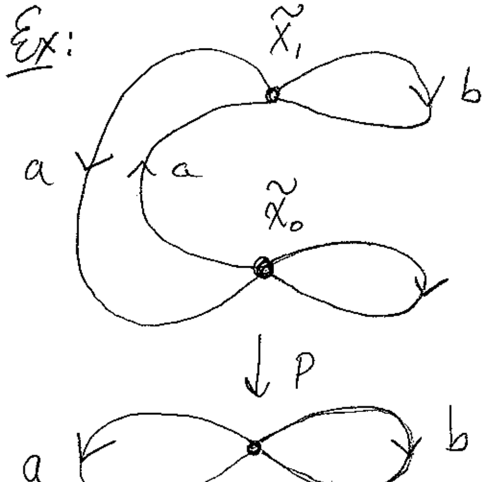
Moreover, if $g \simeq_p f$ and $\tilde{g}$ is its lift starting at $\tilde{x}_0$, then $\tilde{g} \simeq_p \tilde{f}$. In particular $\tilde{g}(1) = \tilde{g}(0)$.

Lifting correspondence: $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ covering map.

$$\Phi : \pi_1(X, x_0) \longrightarrow p^{-1}(x_0)$$

$[f] \longmapsto \tilde{f}(1)$ where $\tilde{f}$ is the lift of f starting at $\tilde{x}_0$.

Ex:



$\Phi([a]) = \tilde{x}_1$

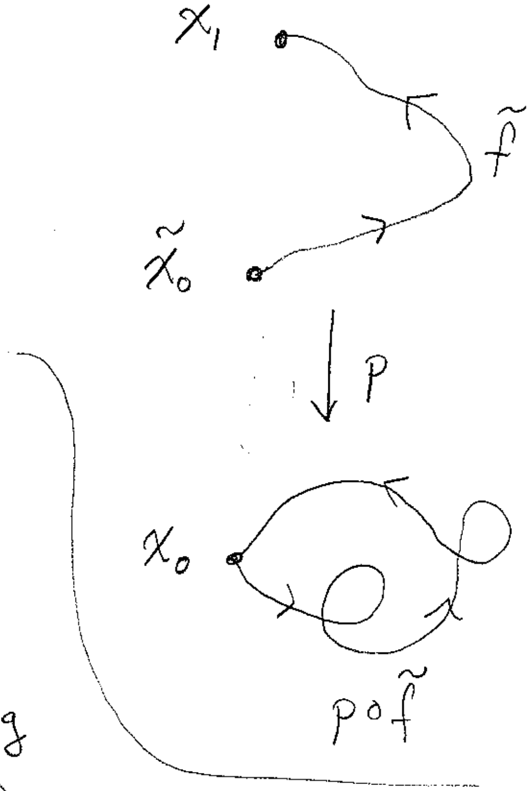
$\Phi([b]) = \tilde{x}_0$

$\Phi([a \cdot b \cdot a]) = \tilde{x}_0$

Thm: If $\tilde{X}$ is simply connected, then Φ is a bijection.

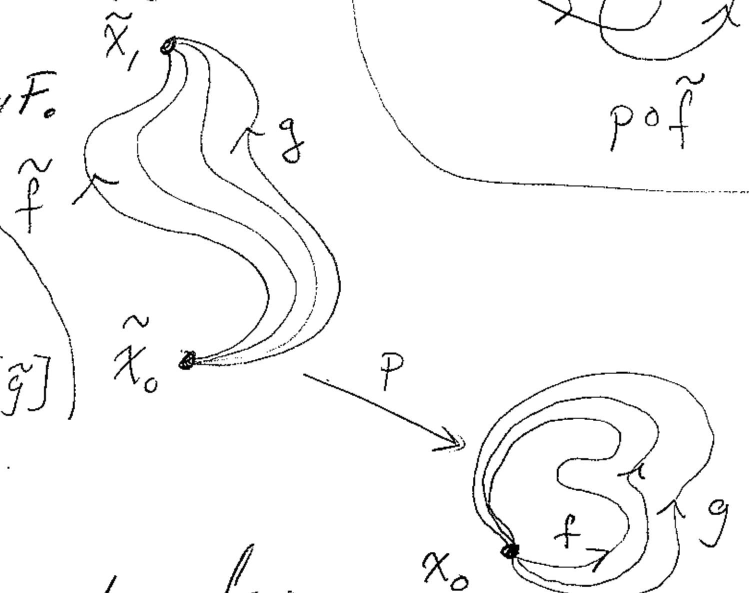
Proof:

<u>Onto</u>: $\tilde{x}_1 \in p^{-1}(x_0)$. As $\tilde{X}$ is path connected, $\exists$ a path $\tilde{f}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Then $[p \circ \tilde{f}] \in \pi_1(X, x_0)$ and $\Phi([p\circ\tilde{f}]) = \tilde{x}_1$.



<u>1-1</u>: Suppose $\Phi([f]) = \Phi([g])$.
Since $\pi_1(\tilde{X}, \tilde{x}_0) = 1$, we have $\tilde{f} \simeq_p \tilde{g}$ via a path homotopy F.

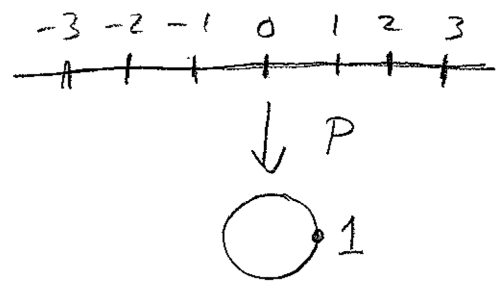
$$\left([\tilde{f}\cdot\bar{\tilde{g}}] = [\text{const } \tilde{x}_0] \text{ and hence } [\tilde{f}] = [\tilde{f}]\cdot\underbrace{[\bar{\tilde{g}}]\cdot[\tilde{g}]}_{\text{const path at } \tilde{x}_1} = [\tilde{g}] \right)$$



Then $p\circ F$ is a path homotopy from f to g, i.e. $[f] = [g]$. ▨

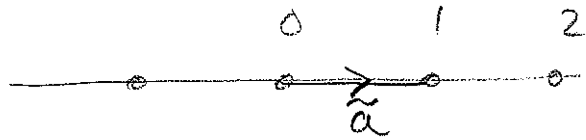
<u>Thm</u>: $\pi_1 S^1 = \mathbb{Z}$

<u>Pf</u>: Consider the usual covering map $p: \mathbb{R} \to S^1$.



<u>Claim</u>: The bijection $\Phi : \pi_1(S^1, 1) \longrightarrow p^{-1}(1) = \mathbb{Z}$ is a group homomorphism (hence isomorphism).

Note $\phi([a^n]) = n$, and



so ϕ is a homomorphism restricted to the cyclic gp $\langle a \rangle$. As

$\Phi(\langle a \rangle)$ is onto, we conclude $\pi_1(S^1, 1) = \langle a \rangle \cong \mathbb{Z}$. ▨

<u>Thm</u>: For $X =$  we have $\pi_1 X = \text{FreeGroup}(a, b)$.

Here:

$$\text{FreeGroup}(a,b) = \left\{ \begin{matrix} \text{words in symbols } a, a^{-1}, b, b^{-1} \\ \text{not containing } aa^{-1}, a^{-1}a, bb^{-1} \text{ or } b^{-1}b \end{matrix} \right\} \cup \{1\}$$

$$= \{a, b, aba^{-1}a^{-1}b^{-1}a, aba^{-1}b^{-1}, \dots\}$$

with group opperation "concatenate + cancel":

$$(aba^{-1}b^{-1}a) \cdot (a^{-1}bba) = aba^{-1}b^{-1}aa^{-1}bba \leftarrow \text{invalid string}$$

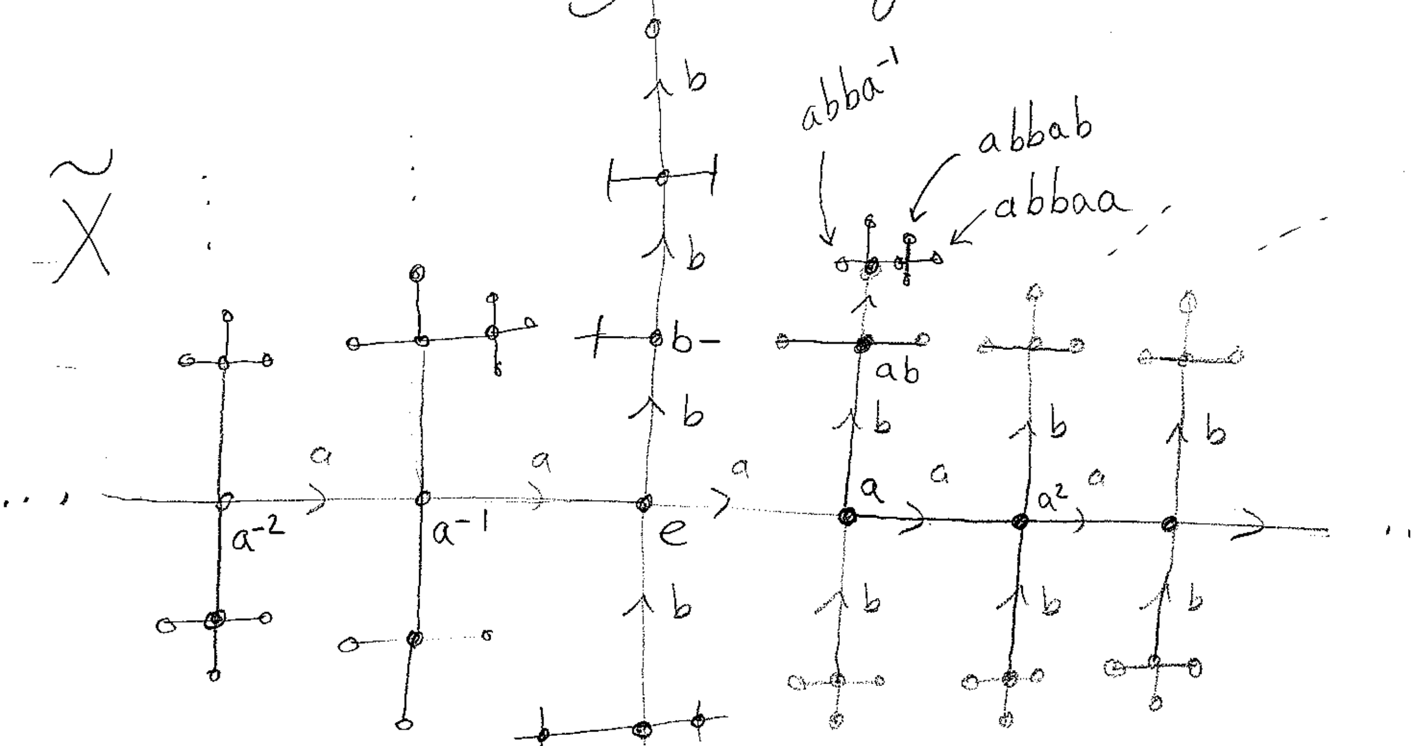
$$= aba^{-1}ba \leftarrow \text{final answer.}$$

Can check associativity, inverses given by reverse and replace $a \leftrightarrow a^{-1}$ $b \leftrightarrow b^{-1}$!

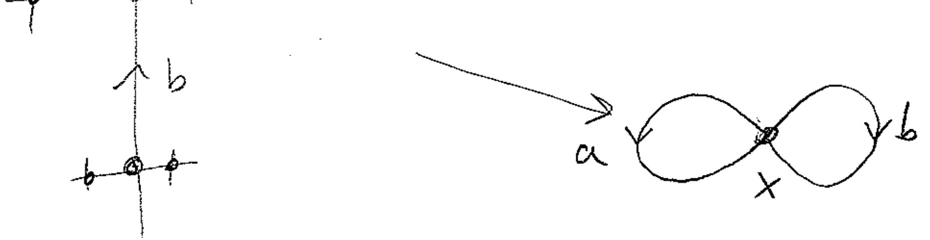
$$(aba^{-1}bb)^{-1} = b^{-1}b^{-1}ab^{-1}a^{-1}$$

"Largest group generated by two elts."

Proof: Consider the covering space of X which is the infinite 4-valent tree with vertices labelled according to the following rule



Note $\tilde{X}$ is simply connected,



and the lifting correspondence with $\tilde{x}_0 = e$ gives a bijection $\Phi : \pi_1(X, x_0) \longrightarrow \text{FreeGroup}(a,b)$

Check: This is a group homomorphism (illustrate with the product of $a^{-1}b$ and ab) $\blacksquare$

- Relate tree structure to assoc of mult in free group
- Formal def of $\tilde{X} = \begin{cases} \text{Vertices} = \text{elts of FreeGroup}(a,b) \\ \text{Edges} = \begin{array}{l} a\text{-edge from } w \text{ to } wa \\ b\text{-edge from } w \text{ to } wb \end{array} \end{cases}$

Next time: Brouwer Fixed Pt. Theorem.