Lecture 7: Applications of π_1

<u>Last time</u>: <u>Thm</u> $\pi_1 S^1 = \mathbb{Z}$, $\pi_1(\infty) = \text{FreeGroup}(a,b)$

[Soon will give Van Kampen's Thm, let us compute π_1.]

<u>Today</u>:

<u>Fund Thm of Algebra</u>: Every non-constant poly $p(z) \in \mathbb{C}[z]$ has a root in $\mathbb{C}$.

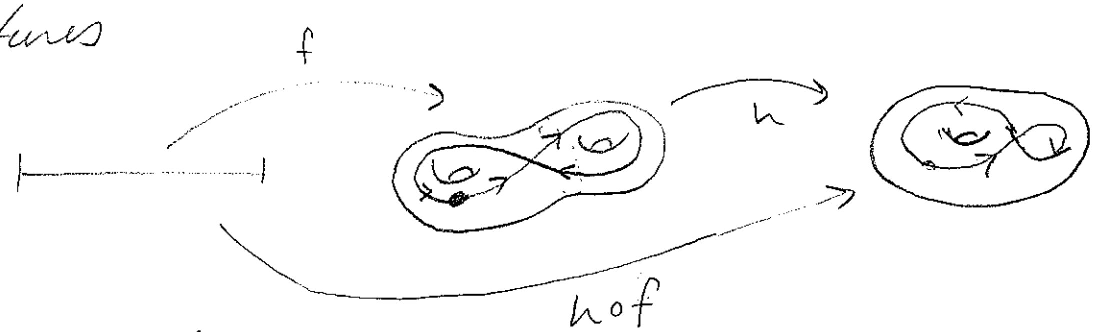
<u>Brouwer Fixed Point Thm</u>: For every cont map $h: D^2 \to D^2$, there is an $x \in D^2$ with $f(x) = x$.

<u>Induced maps</u>: $h: (X, x_0) \to (Y, y_0)$ gives

[Functoriality.] $h_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ via

$$[f] \longmapsto [h \circ f]$$

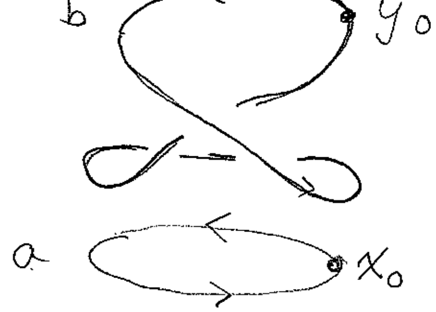
In pictures



h_* is a group homomorphism.

<u>Ex</u>: $h: S^1 \to S^1$

$z \longmapsto z^2$



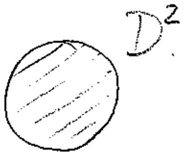
Then

$$h_*: \underset{\langle b \rangle}{\pi_1(S^1, y_0)} \to \underset{\langle a \rangle}{\pi_1(S^1, x_0)}$$

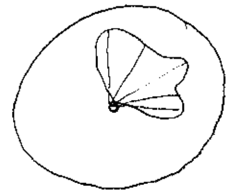
$$b \longmapsto a^2$$

Written additively, $h_*: \mathbb{Z} \to \mathbb{Z}$ is mult by 2.
In general, $z \mapsto z^n$ induces mult by n on $\pi_1 S^1$.

<u>Q</u>: What is $\pi_1(D^2)$, where $D^2 = \{ x \in \mathbb{C} \mid |x| \le 1 \}$?

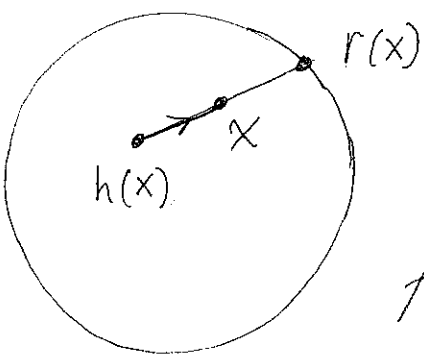


<u>A</u>: Trivial.



Given $f: [0,1] \to D^2$ a loop at $\vec{0}$, set $F(s,t) = t f(s)$.

<u>Proof of B.F.P.T.</u>: Suppose $h: D^2 \to D^2$ has no fixed points. Define $r: D^2 \to S^1$ via the picture at left.

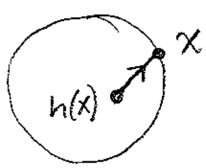


Note r is continuous. [Explicitly, can find a formula for $r(x)$ by noting

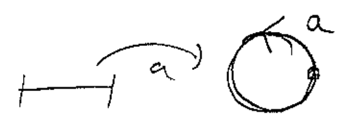
$$r(x) = t(x - h(x)) + x \quad \text{where } t \ge 0$$

is chosen so that $|r(x)| = 1$.]

<u>Key</u>: $r|_{S^1} = id|_{S^1}$

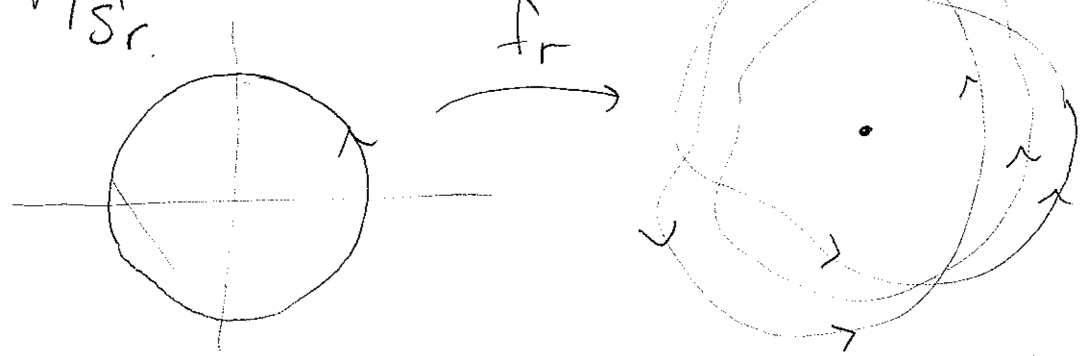


<u>Claim</u>: There does not exist $r: D^2 \to S^1$ with $r|_{S^1} = id|_{S^1}$. [Such an r is called a <u>retract</u> of D^2 to S^1]

Pf 1: Consider the map $S^1 \to S^1$ which is the identity. By assumption, this extends to a map $D^2 \to S^1$. By HW, we know that  must thus be trivial in $\pi_1 S^1$, a contradiction since it generates $\pi_1 S^1 = \mathbb{Z}$.

Pf 2: The map $r_* : \pi_1(D^2, 1) \to \pi_1(S^1, 1)$ has trivial image, since $\uparrow = 0$. If $f : [0,1] \to S^1$ given by $s \mapsto e^{-2\pi s i}$, then $r_*([f]) = [r \circ f]$ $= [id|_{S^1} \circ f] = [f]$ which generates $\pi_1(S^1, 1)$, a contradiction. $\blacksquare$

Proof of the F.T.A.: Let $p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be any polynomial. Let $S^1_r = \{ z \in \mathbb{C} \mid |z| = r \}$, $Y = \mathbb{C} \setminus \{0\}$. For r large have $f_r : S^1_r \to Y$ given by $f_r = p|_{S^1_r}$.



Claim 1: $\pi_1 Y = \mathbb{Z}$, char by winding number.

Claim 2: The map $\pi_1(S^1_r) \to \pi_1(Y)$ is mult by n.

[Argue claim 1 is plausible, as is claim 2 if you consider $a_k = 0$.]

Assume p has no roots. Then

$$\begin{tikzcd} S^1_r \arrow[r, hook, "i"'] \arrow[rr, bend left, "f_r"] & \mathbb{C} \arrow[r, "p"] & Y \end{tikzcd}$$

and so

$$\begin{tikzcd} \overset{\mathbb{Z}}{\pi_1(S^1_r)} \arrow[r, "i_*"] \arrow[rr, bend right, "f_{r*}"'] & \overset{0}{\pi_1(\mathbb{C})} \arrow[r, "p_*"] & \overset{\mathbb{Z}}{\pi_1(Y)} \end{tikzcd}$$

Note that $p_* \circ i_* = f_{r*}$ but the first is the 0 map and the other mult by n. Thus $n = 0$ an p is constant.

Formalization: Consider $f_r : I \to S^1$ given by

$$f_r(s) = \frac{p(re^{-2\pi s i})/p(r)}{|p(re^{-2\pi s i})/p(r)|}$$ which makes sense if p has no zeros.

This is a loop at 1, and equal to 0 in $\pi_1 S^1$ (take $r \to 0$)

O.T.O.H., when r is large, see that $[f] = n[\text{gen}]$.