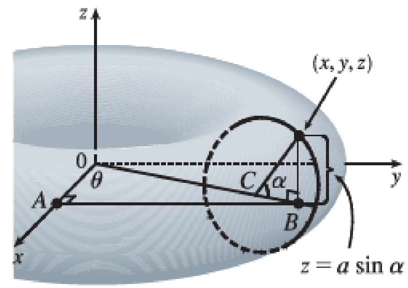
Section 16.5

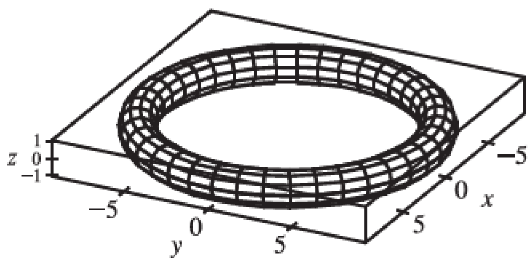
22. $\operatorname{div}\mathbf{F} = \frac{\partial(f(y,z))}{\partial x} + \frac{\partial(g(x,z))}{\partial y} + \frac{\partial(h(x,y))}{\partial z} = \mathbf{0}$ so $\mathbf{F}$ is incompressible.

For Exercises 23–29, let $\mathbf{F}(x,y,z) = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G}(x,y,z) = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$.

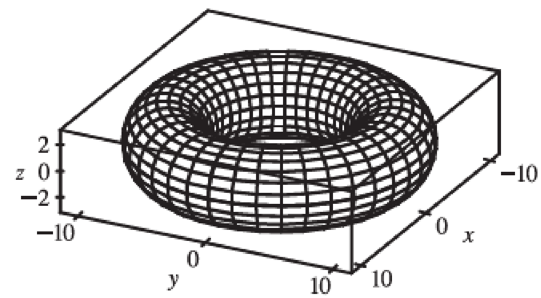
60. (a) Here $z = a\sin\alpha$, $y = |AB|$, and $x = |OA|$. But $|OB| = |OC| + |CB| = b + a\cos\alpha$ and $\sin\theta = \dfrac{|AB|}{|OB|}$ so that $y = |OB|\sin\theta = (b + a\cos\alpha)\sin\theta$. Similarly $\cos\theta = \dfrac{|OA|}{|OB|}$ so $x = (b + a\cos\alpha)\cos\theta$. Hence a parametric representation for the torus is $x = b\cos\theta + a\cos\alpha\,\cos\theta$, $y = b\sin\theta + a\cos\alpha\,\sin\theta$, $z = a\sin\alpha$, where $0 \le \alpha \le 2\pi$, $0 \le \theta \le 2\pi$.



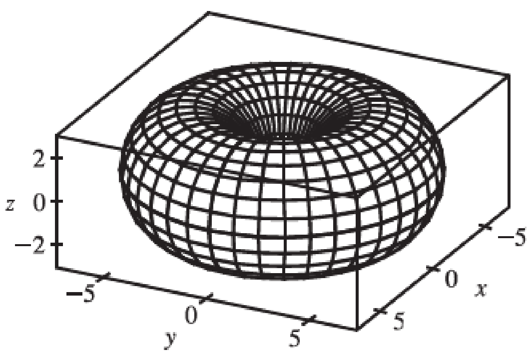
(b)



$a = 1$, $b = 8$



$a = 3$, $b = 8$



$a = 3$, $b = 4$

(c) $x = b\cos\theta + a\cos\alpha\,\cos\theta$, $y = b\sin\theta + a\cos\alpha\,\sin\theta$, $z = a\sin\alpha$, so $\mathbf{r}_\alpha = \langle -a\sin\alpha\,\cos\theta, -a\sin\alpha\,\sin\theta, a\cos\alpha\rangle$, $\mathbf{r}_\theta = \langle -(b + a\cos\alpha)\sin\theta, (b + a\cos\alpha)\cos\theta, 0\rangle$ and

$$\begin{aligned}\mathbf{r}_\alpha \times \mathbf{r}_\theta &= \left(-ab\cos\alpha\,\cos\theta - a^2\cos\alpha\,\cos^2\theta\right)\mathbf{i} + \left(-ab\sin\alpha\,\cos\theta - a^2\sin\alpha\,\cos^2\theta\right)\mathbf{j} \\ &\quad + \left(-ab\cos^2\alpha\,\sin\theta - a^2\cos^2\alpha\,\sin\theta\,\cos\theta - ab\sin^2\alpha\,\sin\theta - a^2\sin^2\alpha\,\sin\theta\,\cos\theta\right)\mathbf{k} \\ &= -a\,(b + a\cos\alpha)\left[(\cos\theta\,\cos\alpha)\,\mathbf{i} + (\sin\theta\,\cos\alpha)\,\mathbf{j} + (\sin\alpha)\,\mathbf{k}\right]\end{aligned}$$

Then $|\mathbf{r}_\alpha \times \mathbf{r}_\theta| = a(b + a\cos\alpha)\sqrt{\cos^2\theta\,\cos^2\alpha + \sin^2\theta\,\cos^2\alpha + \sin^2\alpha} = a(b + a\cos\alpha)$.

Note: $b > a$, $-1 \le \cos\alpha \le 1$ so $|b + a\cos\alpha| = b + a\cos\alpha$. Hence

$A(S) = \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos\alpha)\,d\alpha\,d\theta = 2\pi\left[ab\alpha + a^2\sin\alpha\right]_0^{2\pi} = 4\pi^2 ab$.

Section 16.7

47. Let S be a sphere of radius a centered at the origin. Then $|\mathbf{r}| = a$ and $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3 = (c/a^3)(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$. A parametric representation for S is $\mathbf{r}(\phi,\theta) = a\sin\phi\,\cos\theta\,\mathbf{i} + a\sin\phi\,\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. Then $\mathbf{r}_\phi = a\cos\phi\,\cos\theta\,\mathbf{i} + a\cos\phi\,\sin\theta\,\mathbf{j} - a\sin\phi\,\mathbf{k}$, $\mathbf{r}_\theta = -a\sin\phi\,\sin\theta\,\mathbf{i} + a\sin\phi\,\cos\theta\,\mathbf{j}$, and the outward orientation is given by $\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}$. The flux of $\mathbf{F}$ across S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^\pi \int_0^{2\pi} \frac{c}{a^3}\left(a\sin\phi\,\cos\theta\,\mathbf{i} + a\sin\phi\,\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}\right) \cdot \left(a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}\right) d\theta\,d\phi$$

$$= \frac{c}{a^3}\int_0^\pi\int_0^{2\pi} a^3\left(\sin^3\phi + \sin\phi\,\cos^2\phi\right) d\theta\,d\phi = c\int_0^\pi\int_0^{2\pi}\sin\phi\,d\theta\,d\phi = 4\pi c$$

Thus the flux does not depend on the radius a.

Section 16.8

1. Both H and P are oriented piecewise-smooth surfaces that are bounded by the simple, closed, smooth curve $x^2 + y^2 = 4$, $z = 0$ (which we can take to be oriented positively for both surfaces). Then H and P satisfy the hypotheses of Stokes' Theorem, so by (3) we know $\iint_H \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_P \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ (where C is the boundary curve).

13. The boundary curve C is the circle $x^2 + y^2 = 1$, $z = 1$ oriented in the counterclockwise direction as viewed from above. We can parametrize C by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$, and then $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$. Thus $\mathbf{F}(\mathbf{r}(t)) = \sin^2 t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$, $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = \cos^2 t - \sin^3 t$, and

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}(\cos^2 t - \sin^3 t)\,dt = \int_0^{2\pi}\tfrac{1}{2}(1+\cos 2t)\,dt - \int_0^{2\pi}(1-\cos^2 t)\sin t\,dt$$

$$= \tfrac{1}{2}\left[t + \tfrac{1}{2}\sin 2t\right]_0^{2\pi} - \left[-\cos t + \tfrac{1}{3}\cos^3 t\right]_0^{2\pi} = \pi$$

Now $\operatorname{curl}\mathbf{F} = (1-2y)\,\mathbf{k}$, and the projection D of S on the xy-plane is the disk $x^2 + y^2 \le 1$, so by Equation 17.7.10 [ET 16.7.10] with $z = g(x,y) = x^2 + y^2$ we have

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D (1-2y)\,dA = \int_0^{2\pi}\int_0^1 (1 - 2r\sin\theta)\,r\,dr\,d\theta = \int_0^{2\pi}\left(\tfrac{1}{2} - \tfrac{2}{3}\sin\theta\right)d\theta = \pi.$$

15. The boundary curve C is the circle $x^2 + z^2 = 1$, $y = 0$ oriented in the counterclockwise direction as viewed from the positive y-axis. Then C can be described by $\mathbf{r}(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{k}$, $0 \le t \le 2\pi$, and $\mathbf{r}'(t) = -\sin t\,\mathbf{i} - \cos t\,\mathbf{k}$. Thus $\mathbf{F}(\mathbf{r}(t)) = -\sin t\,\mathbf{j} + \cos t\,\mathbf{k}$, $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = -\cos^2 t$, and $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} -\cos^2 t\,dt = -\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\Big]_0^{2\pi} = -\pi$. Now $\operatorname{curl}\mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$, and S can be parametrized (see Example 17.6.10 [ET 16.6.10]) by $\mathbf{r}(\phi,\theta) = \sin\phi\,\cos\theta\,\mathbf{i} + \sin\phi\,\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$, $0 \le \theta \le \pi$, $0 \le \phi \le \pi$. Then $\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi\,\cos\theta\,\mathbf{i} + \sin^2\phi\,\sin\theta\,\mathbf{j} + \sin\phi\,\cos\phi\,\mathbf{k}$ and

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1} \operatorname{curl}\mathbf{F}\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)\,dA = \int_0^\pi\int_0^\pi\left(-\sin^2\phi\,\cos\theta - \sin^2\phi\,\sin\theta - \sin\phi\,\cos\phi\right)d\theta\,d\phi$$

$$= \int_0^\pi\left(-2\sin^2\phi - \pi\sin\phi\,\cos\phi\right)d\phi = \left[\tfrac{1}{2}\sin 2\phi - \phi - \tfrac{\pi}{2}\sin^2\phi\right]_0^\pi = -\pi$$

16. Let S be the surface in the plane $x + y + z = 1$ with upward orientation enclosed by C. Then an upward unit normal vector for S is $\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$. Orient C in the counterclockwise direction, as viewed from above. $\int_C z\,dx - 2x\,dy + 3y\,dz$ is equivalent to $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F}(x, y, z) = z\,\mathbf{i} - 2x\,\mathbf{j} + 3y\,\mathbf{k}$, and the components of $\mathbf{F}$ are polynomials, which have continuous partial derivatives throughout $\mathbb{R}^3$. We have curl $\mathbf{F} = 3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$, so by Stokes' Theorem,

$$\int_C z\,dx - 2x\,dy + 3y\,dz = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\,dS = \iint_S (3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}) \cdot \tfrac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})\,dS$$
$$= \tfrac{2}{\sqrt{3}} \iint_S dS = \tfrac{2}{\sqrt{3}}\,(\text{surface area of } S)$$

Thus the value of $\int_C z\,dx - 2x\,dy + 3y\,dz$ is always $\frac{2}{\sqrt{3}}$ times the area of the region enclosed by C, regardless of its shape or location. [Notice that because $\mathbf{n}$ is normal to a plane, it is constant. But curl $\mathbf{F}$ is also constant, so the dot product curl $\mathbf{F} \cdot \mathbf{n}$ is constant and we could have simply argued that $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\,dS$ is a constant multple of $\iint_S dS$, the surface area of S.]

19. Assume S is centered at the origin with radius a and let H_1 and H_2 be the upper and lower hemispheres, respectively, of S. Then $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{H_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} + \iint_{H_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$ by Stokes' Theorem. But C_1 is the circle $x^2 + y^2 = a^2$ oriented in the counterclockwise direction while C_2 is the same circle oriented in the clockwise direction. Hence $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = -\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$ so $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$ as desired.

Section 16.9

23. Since $\dfrac{\mathbf{x}}{|\mathbf{x}|^3} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$ and $\dfrac{\partial}{\partial x}\left(\dfrac{x}{(x^2 + y^2 + z^2)^{3/2}}\right) = \dfrac{(x^2 + y^2 + z^2) - 3x^2}{(x^2 + y^2 + z^2)^{5/2}}$ with similar expressions for $\dfrac{\partial}{\partial y}\left(\dfrac{y}{(x^2 + y^2 + z^2)^{3/2}}\right)$ and $\dfrac{\partial}{\partial z}\left(\dfrac{z}{(x^2 + y^2 + z^2)^{3/2}}\right)$, we have

$\operatorname{div}\left(\dfrac{\mathbf{x}}{|\mathbf{x}|^3}\right) = \dfrac{3(x^2 + y^2 + z^2) - 3(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} = 0$, except at $(0, 0, 0)$ where it is undefined.

24. We first need to find $\mathbf{F}$ so that $\iint_S \mathbf{F} \cdot \mathbf{n}\,dS = \iint_S (2x + 2y + z^2)\,dS$, so $\mathbf{F} \cdot \mathbf{n} = 2x + 2y + z^2$. But for S,

$\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Thus $\mathbf{F} = 2\,\mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$ and div $\mathbf{F} = 1$.

If $B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$, then $\iint_S (2x + 2y + z^2)\,dS = \iiint_B dV = V(B) = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$.