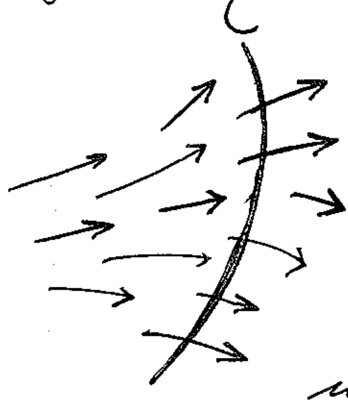
Lecture 36: The divergence thm in $\mathbb{R}^2$ (16.5)

Last time: Flux

$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$



Flux = rate water is crossing $\vec{F}$

$$= \int_C (\vec{F} \cdot \vec{n})\, ds$$

where $\vec{n}$ is a unit normal vector field along C.

$\boxed{\text{Reminder: Exam Wed}}$

[Today, will use the concept of flux to give a different formulation of Green's Thm, one that will explain why it works, and will also generalize to $\mathbb{R}^3$]

Divergence: $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ a vector field with $\vec{F} = (F_1, F_2)$

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

which is [Q.] a function $\mathbb{R}^2 \to \mathbb{R}$

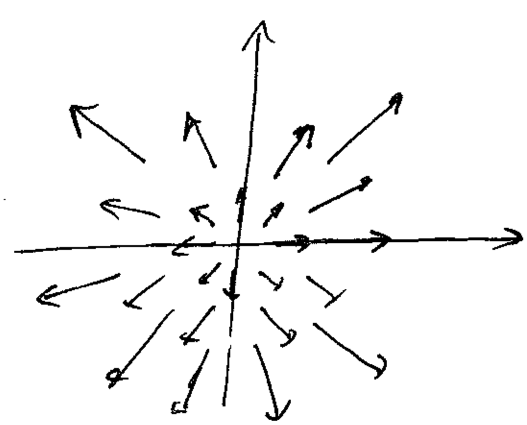
$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

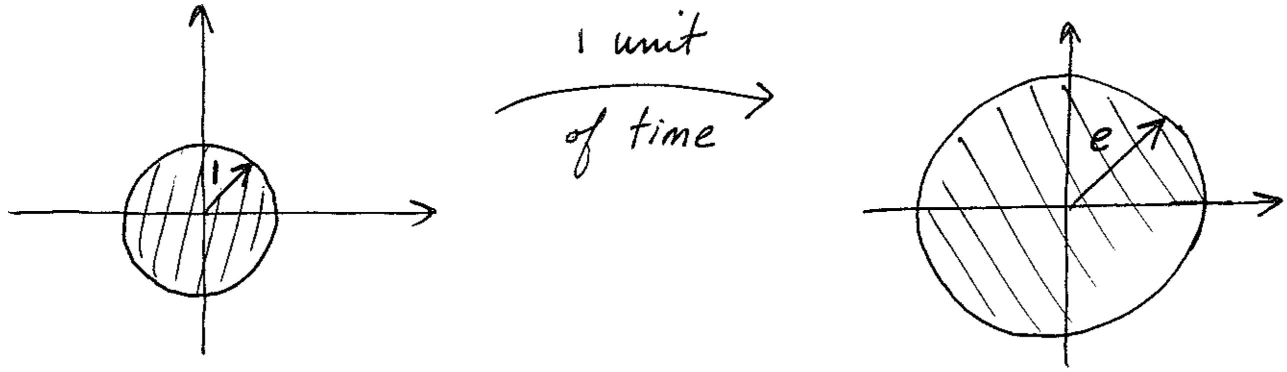
Meaning: [Thinking of $\vec{F}$ as rep. fluid flow:]

$\operatorname{div} \vec{F}$ = rate of expansion of area under the flow.

Ex: $\vec{F} = (x, y)$

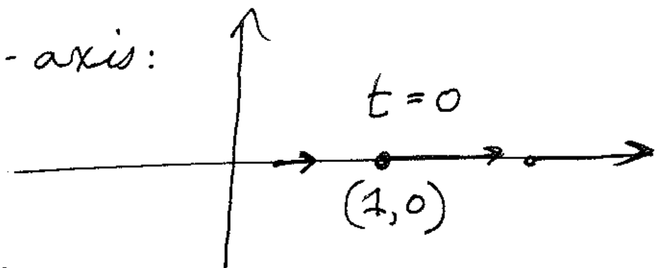
Suppose we dye the water inside the unit circle green.





<u>Reason</u>: Consider the flow on the x-axis:

$x'(t) = x(t) \Rightarrow x(t) = e^t$



In general, the follow of (x_0, y_0) is given by $\vec{f}(t) = (x_0 e^t, y_0 e^t)$ since $\vec{f}'(t) = \vec{f}(t) = \vec{F}(\vec{f}(t))$

So

$$\frac{\text{Green Area}(t)}{\text{Green Area @ } t=0} = \frac{2\pi(e^t)^2}{2\pi} = e^{2t}$$

rate of increase in the green area.

Note

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 2.$$

match!

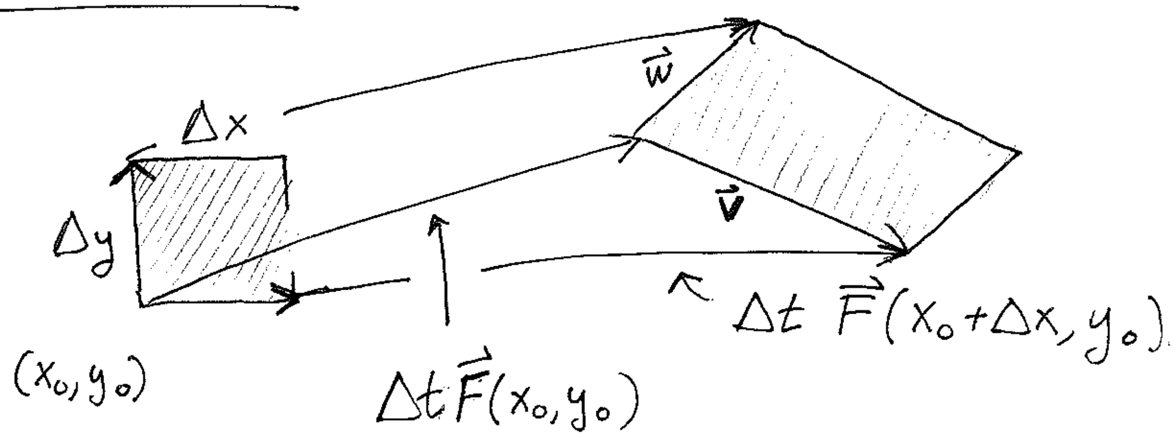
In real life, how could this happen?

(a) Fluid becomes more / less dense (compressible)

[When modeling airflow over a wing, usually assume $\operatorname{div} \vec{F} = 0$ if speeds are $<$ Mach 0.3]

(b) Fluid is being added somehow.

Approx Picture:



$$\vec{v} \approx (\Delta x, 0) + \Delta t \, \Delta x \, \vec{F}_x(x_0,y_0) = \left(\Delta x + \Delta t \, \Delta x \frac{\partial F_1}{\partial x}, \ \Delta t \, \Delta x \frac{\partial F_2}{\partial x}\right)$$

$$= \Delta x \left(1 + \Delta t \frac{\partial F_1}{\partial x}, \ \Delta t \frac{\partial F_2}{\partial x}\right)$$

$$\vec{w} \approx \Delta y \left(\Delta t \frac{\partial F_1}{\partial y}, \ 1 + \Delta t \frac{\partial F_2}{\partial y}\right)$$

consider skipping

So the new area is

$$= \Delta x \Delta y \left(1 + \Delta t (\operatorname{div} \vec{F}) + \Delta t^2 (\text{stuff})\right)$$

Thus:

$$\frac{\text{new area}}{\text{old area}} \approx 1 + \Delta t (\operatorname{div} \vec{F})$$

and so the rate of expansion is $(\operatorname{div} \vec{F})$.

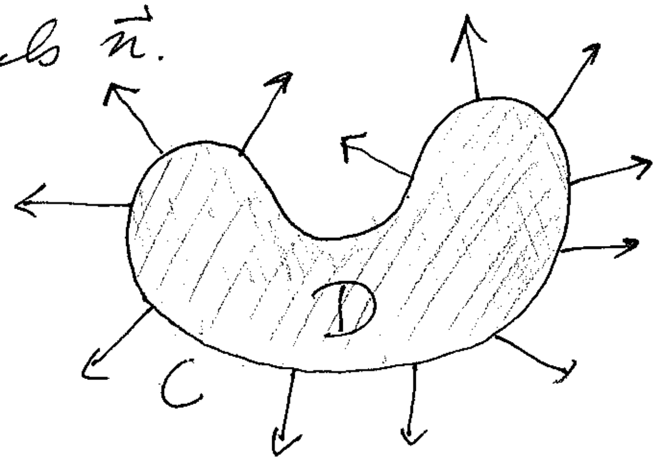
Check units:

$\vec{F}(x,y)$ has units m/s

$\operatorname{div}\vec{F}$ has units 1/s and so $1+\Delta t(\operatorname{div}\vec{F})$ is dimensionless.

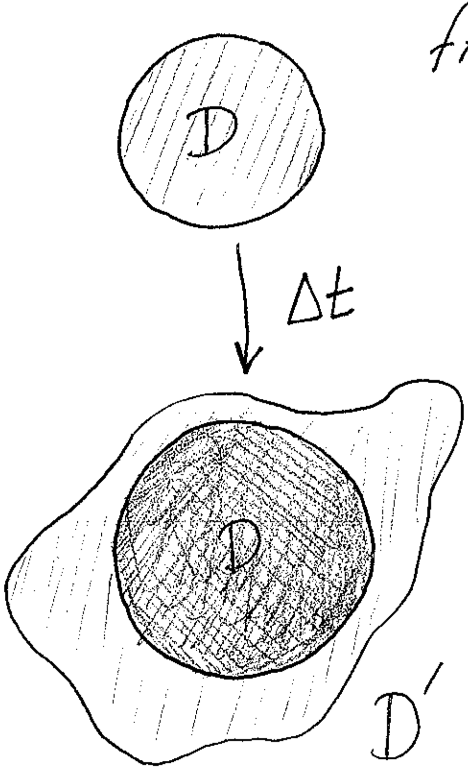
Divergence Thm: D a region in $\mathbb{R}^2$ bounded by C, with outward unit normals $\vec{n}$.

Then



$$\underbrace{\int_C \vec{F}\cdot\vec{n}\, ds}_{\text{Flux}} = \iint_D \operatorname{div}\vec{F}\, dA$$

Reason: After time Δt, the fluid in D now fills the region D'. Now



$$\frac{\text{Area}(D')}{\text{Area}(D)} \approx 1+\Delta t \cdot r$$

Where $r = $ Ave. rate of expansion $= \dfrac{1}{\text{Area}(D)}\iint_D \operatorname{div}\vec{F}\, dA$

$\Longrightarrow$

$$\text{Area}(D') - \text{Area}(D) \approx \Delta t \iint_D \operatorname{div}\vec{F}\, dA$$

Now as the region D is fixed, the change in area can only be accomplished by fluid crossing C. The amount that crosses C in time Δt is $\approx \Delta t \int_C (\vec{F} \cdot \vec{n})\, ds$. Thus we must have $\int_C \vec{F} \cdot \vec{n}\, ds = \iint_D \operatorname{div} \vec{F}\, dA$

How this relates to Green's Thm:

If $\vec{F} = (F_1, F_2)$ set $\vec{G} = (-F_2, F_1) = (P, Q)$

Then

$$\int_C (\vec{F} \cdot \vec{n})\, ds = \int_C \vec{G} \cdot d\vec{r} \qquad \star$$

and

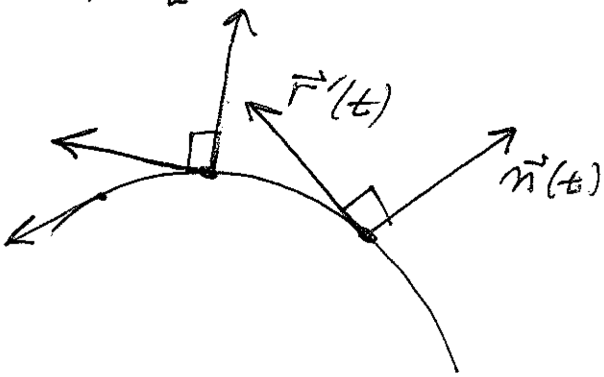
$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

so

$$\iint_D \operatorname{div} \vec{F}\, dA = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

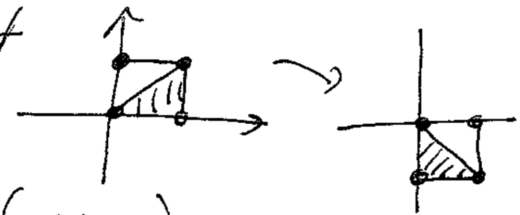
Reason for (*): Take a unit speed param

$\vec{r}:[a,b] \to C$.



Relation between $\vec{r}'(t)$ and $\vec{n}(t)$:

Rotate right



$T(u,v) = (-v, u)$

So $\vec{n}(t) = (-r_2'(t), r_1'(t))$ and

$$\int_C \vec{F} \cdot \vec{n}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{n}(t)\, dt$$

$$= \int_a^b - F_1(\vec{r}(t)) \cdot r_2'(t) + F_2(\vec{r}(t)) \cdot r_1'(t)\, dt$$

$$= \int_a^b \vec{G}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_C \vec{G} \cdot d\vec{r}$$

Next time: Flux across a surface