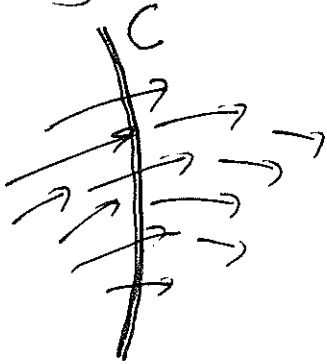


# Lecture 38: Flux and the Divergence Theorem in $\mathbb{R}^3$ . (16.7, 16.9) 110

Note: HW for today due Wed, Dec 1.

Previously:  $\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  Flux = rate fluid crosses  $C$

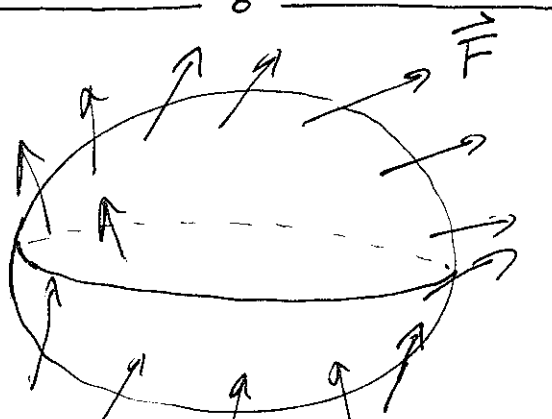


$$= \int_C (\vec{F} \cdot \vec{n}) ds$$

where  $\vec{n}$  is a unit normal vector field.

$$\vec{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$S$  a surface



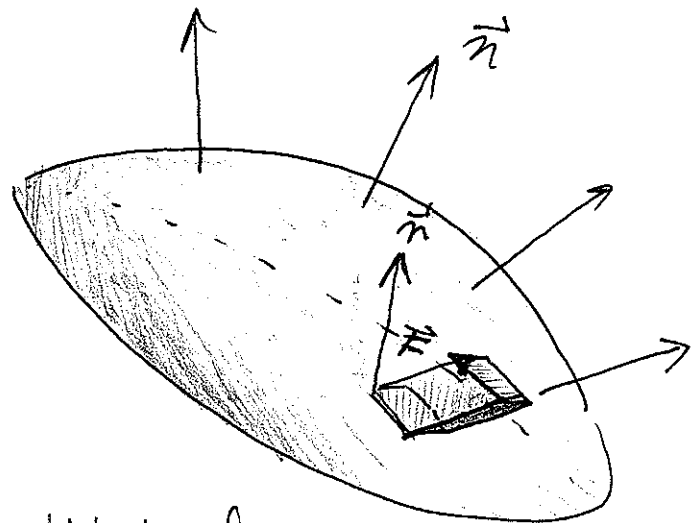
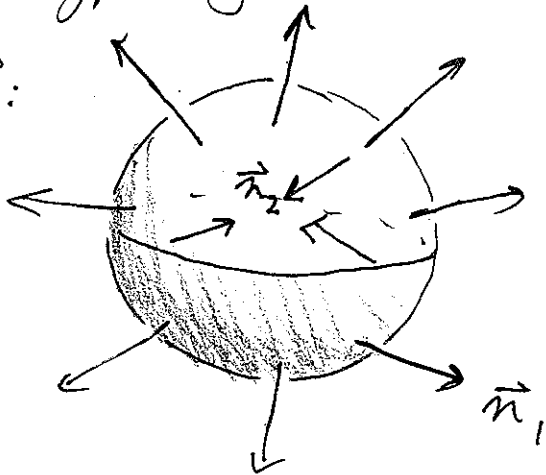
$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n} dA = \iint_S \vec{F} \cdot d\vec{S}$$

↑  
Notation

where  $\vec{n}$  is a unit normal vector field.

Note: Typically two choices

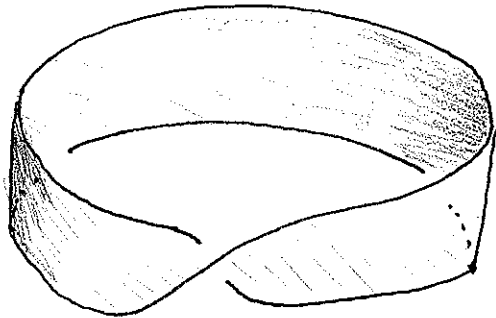
for  $\vec{n}$ :



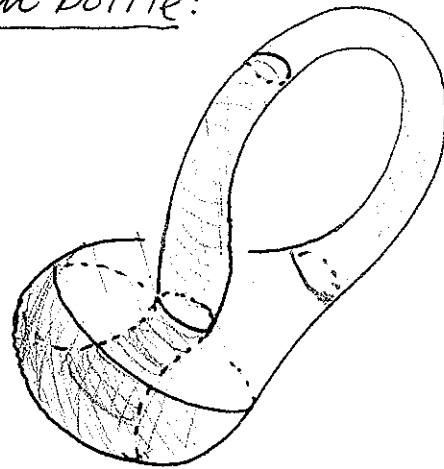
Works for same reason as before.

Some surfaces have no good choice of normal vector,  
 so can't integrate vector fields on these:

Möbius band:



Klein bottle:



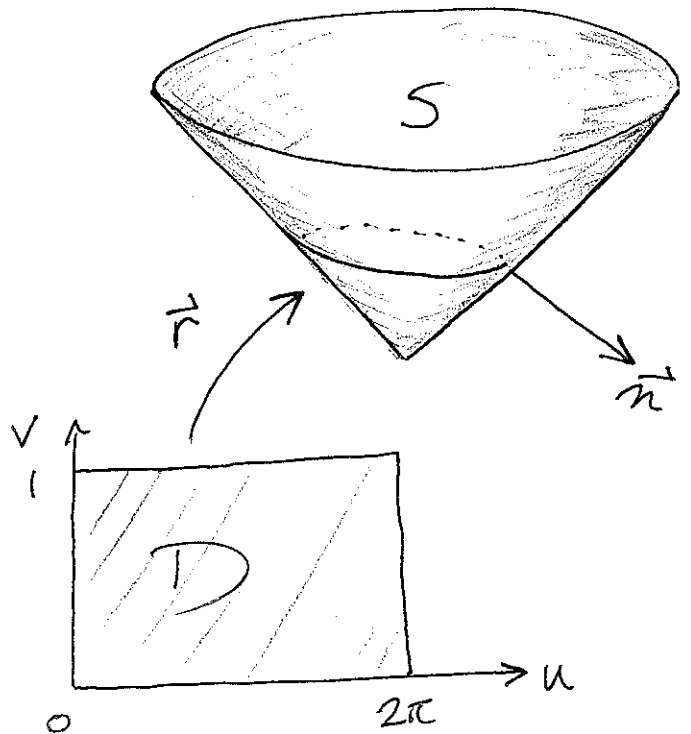
[Show physical models.]

Ex: Cone  $x^2 + y^2 = z^2$   $0 \leq z \leq 1$

$$\vec{F} = (z, 1, x)$$

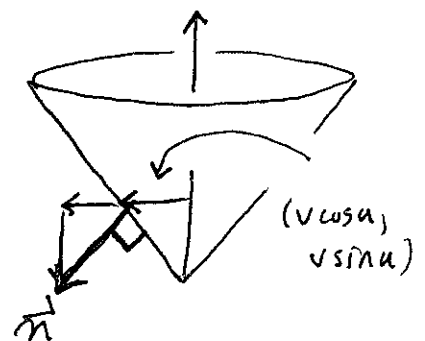
Compute:  $\iint_S (\vec{F} \cdot \vec{n}) dA$

$$\vec{r}(u, v) = (v \cos u, v \sin u, v)$$



$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \frac{(v \cos u, v \sin u, -v)}{\sqrt{v^2 \cos^2 u + v^2 \sin^2 u + v^2}} \leftarrow \text{from earlier}$$

$$= \frac{1}{\sqrt{2}} (\cos u, \sin u, -1)$$



$$\underbrace{\vec{F}(\vec{r}(u,v)) \cdot \vec{n}} = \frac{1}{\sqrt{2}} (v \cos u + \sin u - v \cos u)$$

$$(v, 1, v \cos u) = \frac{1}{\sqrt{2}} \sin u$$

$$S_0 \iint_S (\vec{F} \cdot \vec{n}) dA = \int_0^1 \int_0^{2\pi} \left( \frac{1}{\sqrt{2}} \sin u \right) \cdot \overbrace{|\vec{r}_u \times \vec{r}_v|}^{\sqrt{2}v} du dv$$

$$= \int_0^1 \int_0^{2\pi} v \sin u du dv = 0$$

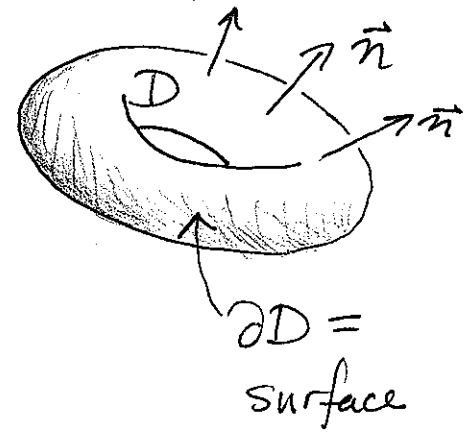
Short cut:

$$\iint_S (\vec{F} \cdot \vec{n}) dA = \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \cdot |\vec{r}_u \times \vec{r}_v| du dv$$

$$= \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) du dv$$

Divergence Thm:  $D$  a region in  $\mathbb{R}^3$ ,  $\vec{F}$  a vector field  
 $\vec{n}$  outward pointing normal.

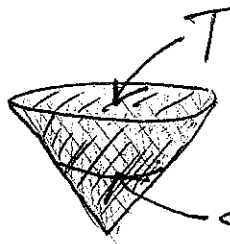
$$\iint_{\partial D} (\vec{F} \cdot \vec{n}) dA = \iiint_D \text{div } \vec{F} dV$$



Here, if  $\vec{F} = (F_1, F_2, F_3)$  then

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad \left( \begin{array}{l} \text{a fn from} \\ \mathbb{R}^3 \rightarrow \mathbb{R} \end{array} \right)$$

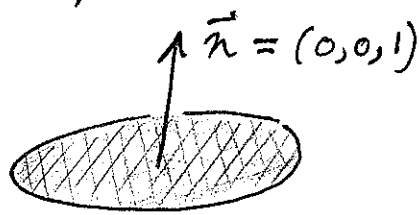
Everything can be interpreted as before.

Ex:  $D =$   = solid =  $z^2 \geq x^2 + y^2$   
 $z \leq 1$

$$\vec{F} = (z, 1, x)$$

$$\iint_{\partial D} (\vec{F} \cdot \vec{n}) dA = \underbrace{\iint_S (\vec{F} \cdot \vec{n}) dA}_0 + \iint_T (\vec{F} \cdot \vec{n}) dA$$

$$\iint_T (\vec{F} \cdot \vec{n}) dA = \iint_T x dA = 0 \quad \text{by symmetry.}$$



$$\text{So } \iint_{\partial D} (\vec{F} \cdot \vec{n}) dA = \boxed{0}$$

Compare:

$$\iiint_D \operatorname{div} \vec{F} dV = \iiint_D 0 dV = \boxed{0} \quad \checkmark$$

$$\frac{\partial}{\partial x} z + \frac{\partial}{\partial y} 1 + \frac{\partial}{\partial z} x = 0$$

Application: Heat flow.

$u(x, y, z, t)$  = temperature at  $(x, y, z)$  at time  $t$ .

$$\frac{\partial u}{\partial t} = c \Delta u = c \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

↑ Laplacian

Will now explain why.

Newton's Law of Cooling:

hot to cold

$$\text{Heat flow} = -k \text{ grad}(u) = -k \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right)$$

Rate heat flows into a region  $D$ :

$$\iint_{\partial D} (-k \text{ grad}(u)) \cdot \vec{n} \, dA = \iiint_D k \text{ div}(\text{grad} u) \, dV$$

↓ inward normal

$$= \iiint_D k \Delta u \, dV$$

Amount of heat energy in  $D$ :

$$\iiint_D \sigma \rho u \, dV$$

$\sigma$  - specific heat } constants  
 $\rho$  - mass density }

and the rate it is changing is

$$\frac{\partial}{\partial t} \iiint_D \sigma \rho u(x, y, z, t) dx dy dz$$

$$= \iiint_D \sigma \rho \frac{\partial u}{\partial t} dV$$

$$= \iiint_D k \Delta u dV$$

As this is true for all regions, must have

$$k \Delta u = \sigma \rho \frac{\partial u}{\partial t} \Rightarrow \frac{\partial u}{\partial t} = \frac{k}{\sigma \rho} \Delta u.$$

Unit check: Heat energy is in J, so flux is J/s = W

$$k \text{ in } \frac{\text{W}}{\text{mK}}, \quad \rho \text{ in } \frac{\text{g}}{\text{m}^3}, \quad \sigma = \frac{\text{J}}{\text{gK}}.$$