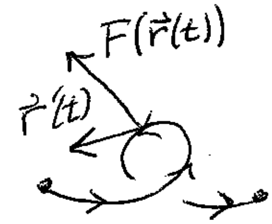
Lecture 21: More on integrating vector fields (16.2)

Last time:



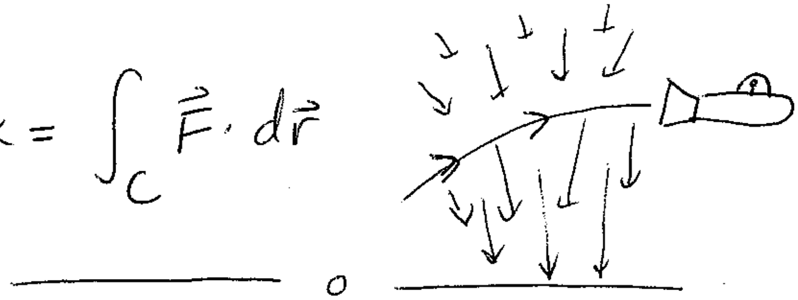
C oriented curve in $\mathbb{R}^n$

$\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ a vector field

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

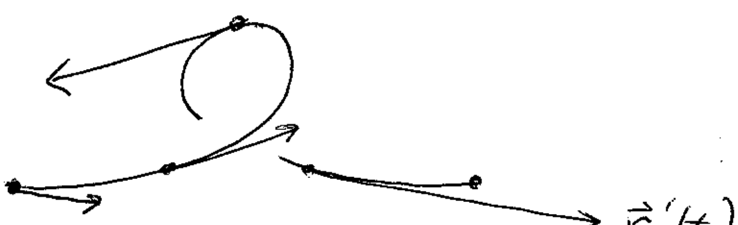
for any parameterization $\vec{r}: [a,b] \to \mathbb{R}^n$ of C.

Ex: $\text{Work} = \int_C \vec{F} \cdot d\vec{r}$



[Other examples from E+M see Ampère's Law on HW.]

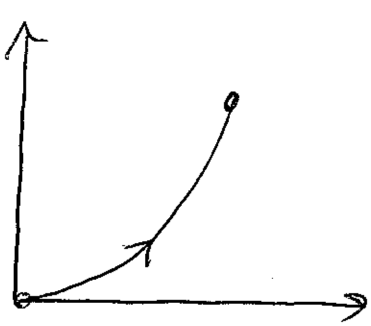
Alternate notation:



Unit Tangent vectors

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Ex:



$\vec{r}(t) = (t, t^2)$

$\vec{r}'(t) = (1, 2t) \qquad |\vec{r}'(t)| = \sqrt{1+4t^2}$

$\vec{T}(t) = \left(\frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}}\right)$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \vec{F}(\vec{r}(t))\cdot \underbrace{\left(\frac{\vec{r}\,'(t)}{|r'(t)|}\right)}_{\vec{T}(t)} \underbrace{|\vec{r}\,'(t)|\,dt}_{ds}$$

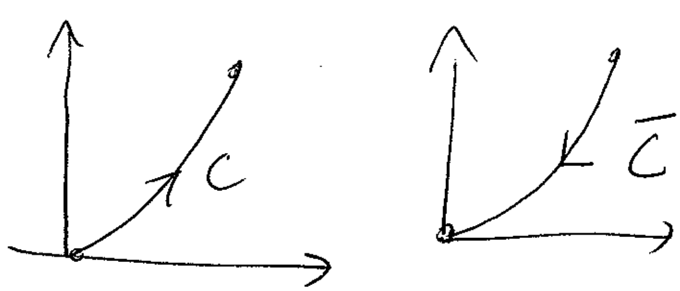
$$= \int_C \vec{F}\cdot\vec{T}\,ds \longleftarrow \left[\begin{array}{l}\text{integral of a fn w.r.t.}\\ \text{arc length}\end{array}\right]$$

Note that ↗ doesn't change if we use a different parameterization, unless we travel the other way, $\vec{T}(t) \longleftrightarrow -\vec{T}(t)$ and so and then

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds = -\int_C \vec{F}\cdot(-\vec{T})\,ds$$

$$= -\int_{\bar{C}} \vec{F}\cdot\vec{T}\,ds = -\int_{\bar{C}} \vec{F}\cdot d\vec{r}.$$

C oriented the other way



More notation:

C curve in $\mathbb{R}^2$, parameterized by $\vec r : [a,b] \to \mathbb{R}^2$

$$\vec r(t) = (x(t), y(t)) = x(t)\vec i + y(t)\vec j$$

$$\vec F : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \vec F(x,y) = P(x,y)\vec i + Q(x,y)\vec j$$

Then

$P(\vec r(t))\vec i + Q(\vec r(t))\vec j$

$x'(t)\vec i + y'(t)\vec j$

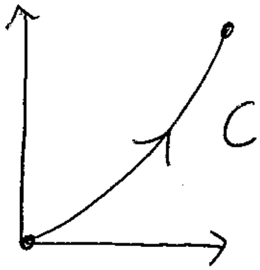
$$\int_C \vec F\, d\vec r\,'(t) = \int_a^b \vec F(\vec r(t)) \cdot \vec r\,'(t)\, dt$$

$$= \int_a^b \Big(P(\vec r(t))x'(t) + Q(\vec r(t))y'(t)\Big)\, dt$$

$$= \int_a^b P(\vec r(t)) \underbrace{x'(t)\, dt}_{dx} + \int_a^b Q(\vec r(t)) \underbrace{y'(t)\, dt}_{dy}$$

$$= \int_C P\, dx + \int_C Q\, dy =$$

$$= \int_C P\, dx + Q\, dy$$



$$\int_C -y\,dx + x\,dy$$

$\vec{r}(t) = (\overset{x}{\overset{\shortparallel}{t}}, \overset{y}{\overset{\shortparallel}{t^2}})$

$0 \leq t \leq 1$

$$= \int_0^1 -t^2 \overbrace{(1\,dt)}^{x'(t)dt} + t\overbrace{(2t\,dt)}^{y'(t)\,dt}$$

$$= \int_0^1 t^2\,dt = \frac{1}{3}.$$

[Same as example at the end of last time.]

Similarly in $\mathbb{R}^3$ $\displaystyle\int_C P\,dx + Q\,dy + R\,dz$

$$= \int_C \vec{F}\cdot d\vec{r} \quad \text{where } \vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}.$$

Recall:

Fundamental Thm of Calc: $f: [a,b] \to \mathbb{R}$ differentiable.

Then $\displaystyle\int_a^b f'(t)\,dt = f(b) - f(a)$

~~Compare: Fundamental Thm of Line integrals:~~

~~$f: \mathbb{R}^n \to \mathbb{R}$ differentiable. If C is a curve in $\mathbb{R}^n$,~~

~~then $\int_C \nabla f \cdot d\vec{r} =$~~

Fund. Thm of Line Integrals:

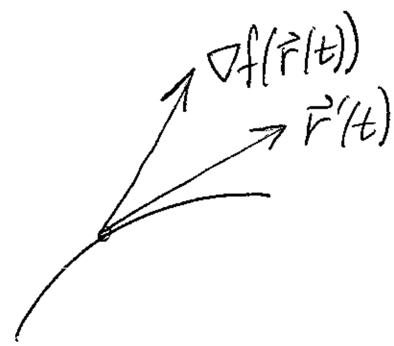
$f: \mathbb{R}^n \to \mathbb{R}$ diff fn.

C a curve in $\mathbb{R}^n$ from A to B. Then

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

Reason: Pick $\vec{r}: [a,b] \to C$ where $\vec{r}(a) = A$, $\vec{r}(b) = B$.

Then



$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

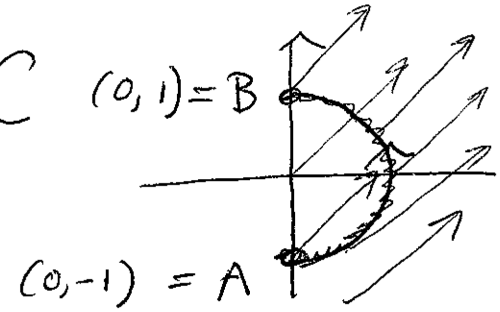
$$= \int_a^b \left(D_{\vec{r}'(t)} f\right)(\vec{r}(t))\, dt$$

rate f is changing as we move along the curve at time t.

$$= \int_a^b \left(\begin{matrix}\text{rate of} \\ \text{change in } f(t)\end{matrix}\right) dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)) = f(B) - f(A).$$

<u>Ex:</u> $f(x,y) = x+y$ $\quad C$



$\vec{F} = \nabla f = (1,1)$

$\int_C \vec{F} \cdot d\vec{r}$ $\qquad \vec{r}(t) = (\cos t, \sin t)$ for $-\pi/2 \le t \le \pi/2$

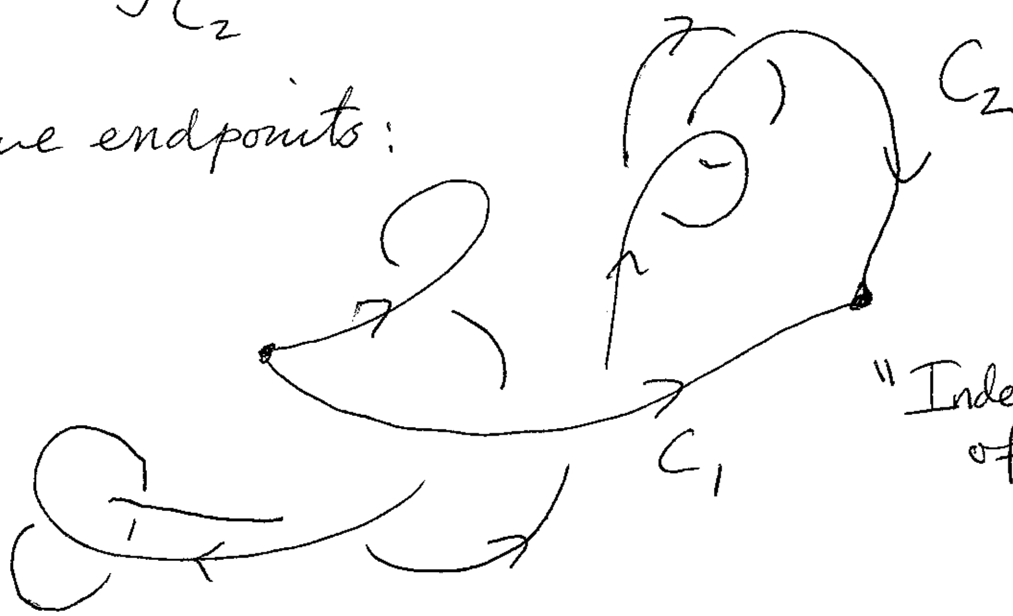
$$= \int_{-\pi/2}^{\pi/2} (1,1) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_{-\pi/2}^{\pi/2} -\sin t + \cos t \; dt = \cos t + \sin t \Big|_{t=-\pi/2}^{t=\pi/2}$$

$$= 1 - (-1) = 2 = f(0,1) - f(0,-1) \checkmark$$

<u>Consequence:</u> If $\vec{F} = \nabla f$ then

$\int_{C_1} F \cdot d\vec{r} = \int_{C_2} F \cdot d\vec{r}$ if C_1 and C_2 have the same endpoints:



"Independence of path."