Lecture 11: The chain rule (14.5) and directional derivatives (14.6).

Last time: Chain rule

① $f: \mathbb{R}^2 \to \mathbb{R}$, $x, y: \mathbb{R} \to \mathbb{R}$

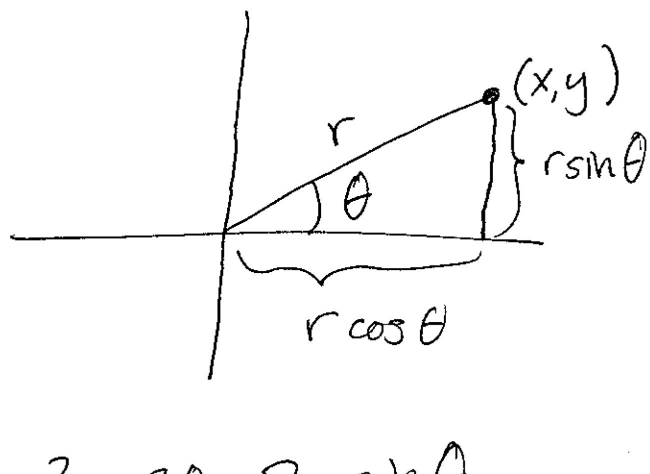
Consider $h(t) = f(x(t), y(t))$. Then

$$h'(t) = \frac{\partial f}{\partial x}(x(t), y(t)) \cdot x'(t) + \frac{\partial f}{\partial y}(x(t), y(t))\, y'(t)$$

② $z = f(x,y)$ with $x = x(t)$ and $y = y(t)$. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

What if x and y themselves depend on more than one var?



Ex: $z = f(x,y) = x^2 + 3y$

$x(r,\theta) = r\cos\theta$
$y(r,\theta) = r\sin\theta$

So $z(r,\theta) = f(x(r,\theta), y(r,\theta)) = r^2\cos^2\theta + 3r\sin\theta$

Chain Rule: $$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= 2x \cdot (-r\sin\theta) + 3 \cdot (r\cos\theta)$$

$$= -2r^2 \sin\theta\cos\theta + 3r\cos\theta$$

[Easy to see this matches what we would get by doing it directly.]

<u>Ex</u>: $u = f(x, y, z)$

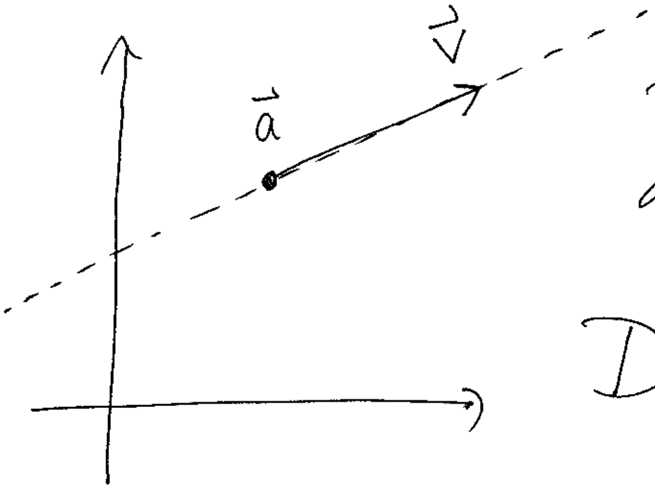
$$x = x(s,t)$$
$$y = y(s,t)$$
$$z = z(s,t)$$

<u>Chain Rule</u>: $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial s} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial s}$

<u>Directional Derivatives</u>: (14.6) $f : \mathbb{R}^2 \to \mathbb{R}$

[Already have ∂-derivatives, measuring change in (x,y) directions. But why give axes special treatment?]

Pick a point $\vec{a}$ in $\mathbb{R}^2$ and a direction $\vec{v}$.



The derivative of f in direction $\vec{v}$ at $\vec{a}$ is

$$D_{\vec{v}} f(\vec{a}) = \text{rate of change in } f \text{ as we move in direction } \vec{v} \text{ away from } \vec{a}.$$

$$\# \cancel{\lim_{t \to 0}} = \frac{d}{dt} \underbrace{f(\vec{a} + t\vec{v})}_{\text{fn of var}} \Big|_{t=0}$$

Ex: If $\vec{v} = \vec{i}$, then $D_{\vec{i}} f(\vec{a}) = \frac{\partial f}{\partial x}(\vec{a})$

In general can find via the chain rule:

$$\vec{a} = (a_1, a_2) \qquad \vec{a} + t\vec{v} = (a_1 + tv_1, \, a_2 + tv_2)$$
$$\vec{v} = (v_1, v_2)$$

$$f(\vec{a} + t\vec{v}) = f(x, y) \text{ where } x = a_1 + tv_1, \; y = a_2 + tv_2$$

$$\cancel{f'(0) = \frac{\partial f}{\partial x}(a_1, a_2) \frac{dx}{dt}}$$

Now:

$$f'(0) = \frac{\partial f}{\partial x}(x(0),y(0)) \cdot x'(0) + \frac{\partial f}{\partial y}(x(0),y(0)) \cdot y'(0)$$

$$= \frac{\partial f}{\partial x}(a_1,a_2)\, v_1 + \frac{\partial f}{\partial y}(a_1,a_2)\, v_2$$

$$= D_{\vec{v}} f(\vec{a}).$$

Ex: $f(x,y) = x^2+y^2$

$\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$

Unit vector: Usually want to take dir. dirivatives with resp. to unit vectors. [Book insists on this.]

$D_{\vec{u}} f(2,1)$

$$= \frac{\partial f}{\partial x}(2,1) \cdot \frac{1}{\sqrt{2}} + \frac{\partial f}{\partial y}(2,1) \cdot \frac{1}{\sqrt{2}}$$

$$= 4 \cdot \frac{1}{\sqrt{2}} + 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}(3) = 3\sqrt{2}.$$

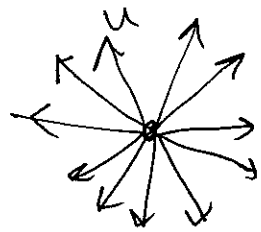
<u>Gradient</u>: $f : \mathbb{R}^2 \to \mathbb{R}$

$$\nabla f(\vec{a}) = \left(\frac{\partial f}{\partial x}(\vec{a}), \frac{\partial f}{\partial y}(\vec{a}) \right)$$

[Will give a geom. interp. in a minute, but for now:]

$$D_{\vec{v}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$$

Suppose we want to know: in which direction does f increase fastest?

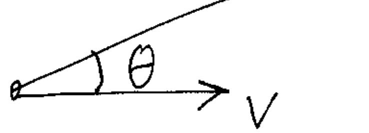


Suppose $\vec{v}$ is a unit vector.

Then

$$D_{\vec{v}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v} = |\nabla f(\vec{a})| \cos \theta$$

So to maximize, want $\theta = 0$,



i.e. $\vec{v}$ points in the same dir as ∇f, and so

$$\vec{V} = \frac{\nabla f(\vec{a})}{|\nabla f(\vec{a})|}$$

<u>Thus</u>: $\nabla f(\vec{a})$ points in direction of fastest increase of f. Length is rate of said increase.

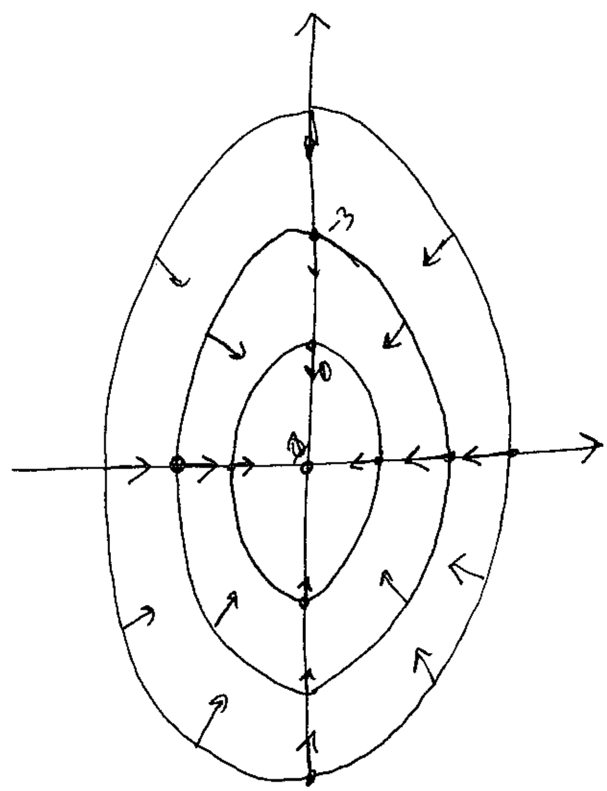
<u>Ex</u>: $f(x,y) = 1 - 4x^2 - y^2$

$\nabla f = (-8x, -2y)$

Level sets:

$f=0$: $4x^2+y^2=1$

$f=-3$: $4x^2+y^2=4$



What is $\nabla f(\vec{0})$?

<u>A</u>: $\vec{0}$.

<u>Moral</u>: A min/max can only occur when $\nabla f = \vec{0}$.