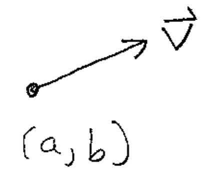
Lecture 12: Gradient (14.6)

Reminder: Exam Wed



Last time: $f: \mathbb{R}^2 \to \mathbb{R}$

$D_{\vec{v}} f(a,b) =$ rate f changes as we go in direction $\vec{v}$ starting at (a,b)

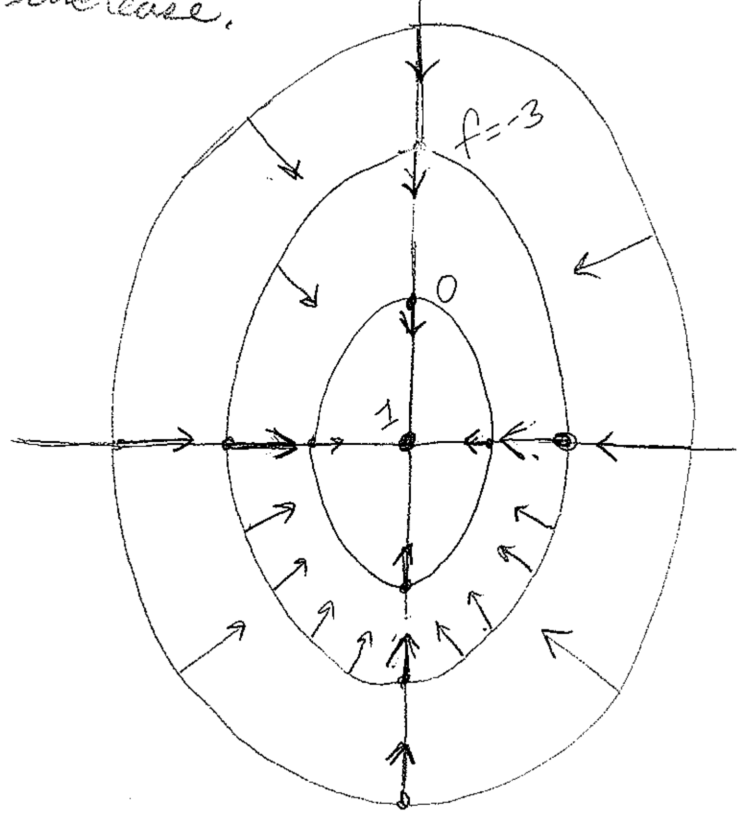
$$\nabla f(a,b) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right)$$

Reationship: $D_{\vec{v}} f(a,b) = \vec{v} \cdot \nabla f(a,b)$

Meaning: $\nabla f(a,b)$ points in direction of fastest increase in f at (a,b). $|\nabla f(a,b)|$ is the rate of that increase.

Ex: $f(x,y) = 1 - 4x^2 - y^2$

$\nabla f(x,y) = (-8x, -2y)$



Level sets:

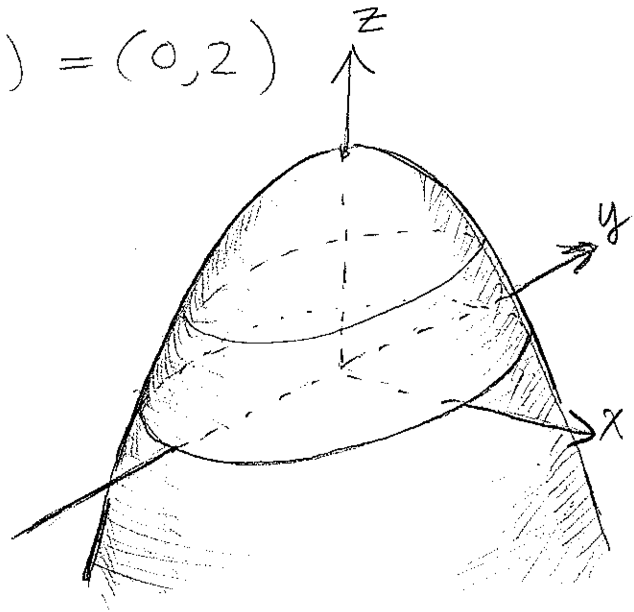
$f = 1$: $(0,0)$

$f = 0$: $4x^2 + y^2 = 1$

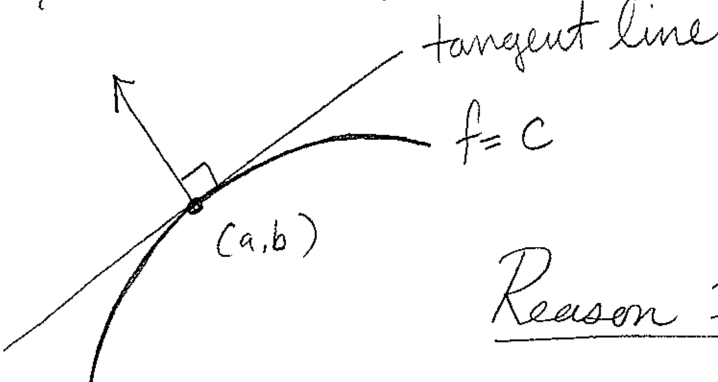
$f = -3$: $4x^2 + y^2 = 4$

$\nabla f(1,0) = (-8,0)$ $\quad$ $\nabla f(-1,0) = (8,0)$

$\nabla f(0,1) = (0,-2)$ $\quad$ $\nabla f(0,-1) = (0,2)$

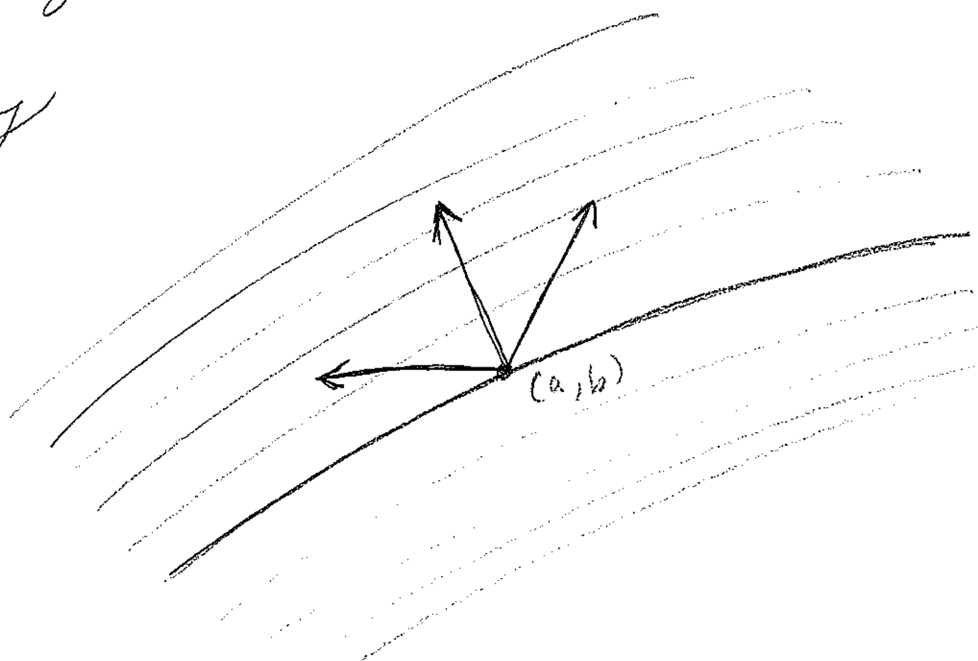


<u>Key</u>: $\nabla f(a,b)$ is at right angles to the level set of f containg (a,b).

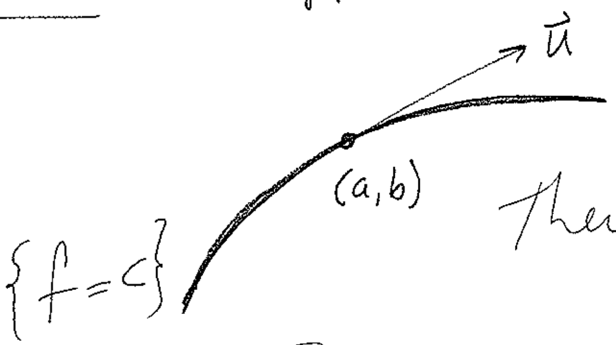


<u>Reason 1</u>: Gradient points in direction of fastest increase in f.

To cross as many level sets as possible with a vector of fixed length, should go at right angles to the level sets.



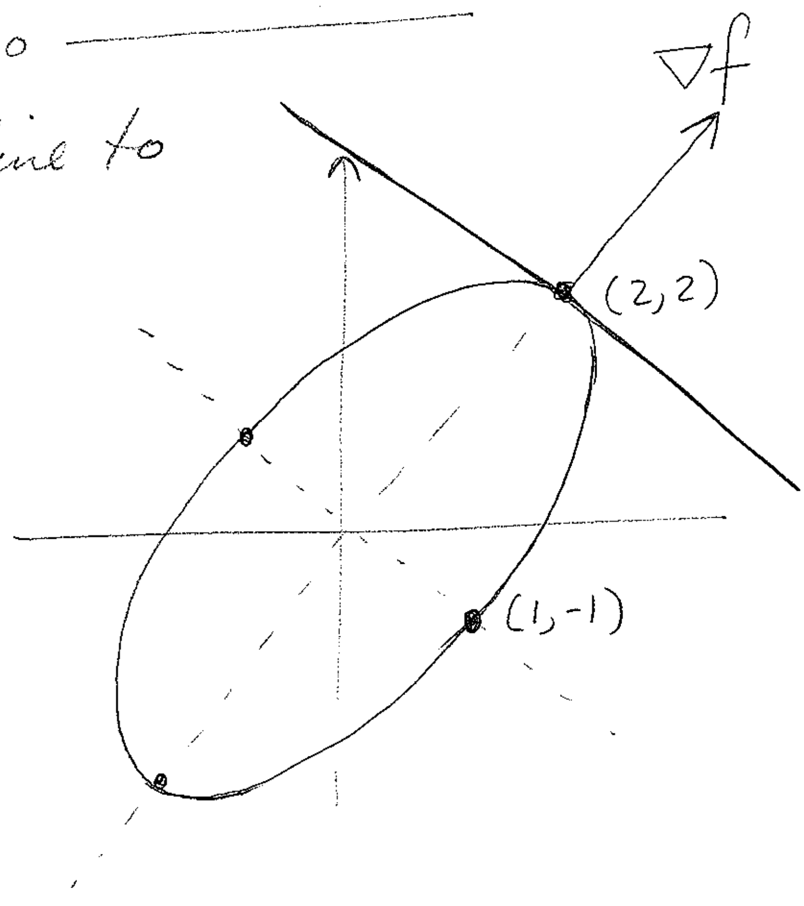
Reason 2: Suppose $\vec{u}$ is tangent to the level set at (a,b).



then $D_{\vec{u}} f(a,b) = 0$

So $0 = \vec{u} \cdot \nabla f(a,b) \Longrightarrow$

$\nabla f(a,b)$ is $\perp$ to $\vec{u}$.



Ex: Find the tangent line to the ellipse C given by:

$$5x^2 + 5y^2 - 3xy = 8$$

at $(2,2)$

A. View C as the level set $f = 8$ for

$$f(x,y) = 5x^2 + 5y^2 - 3xy$$

$$\nabla f = (10x - 3y,\ 10y - 3x)$$

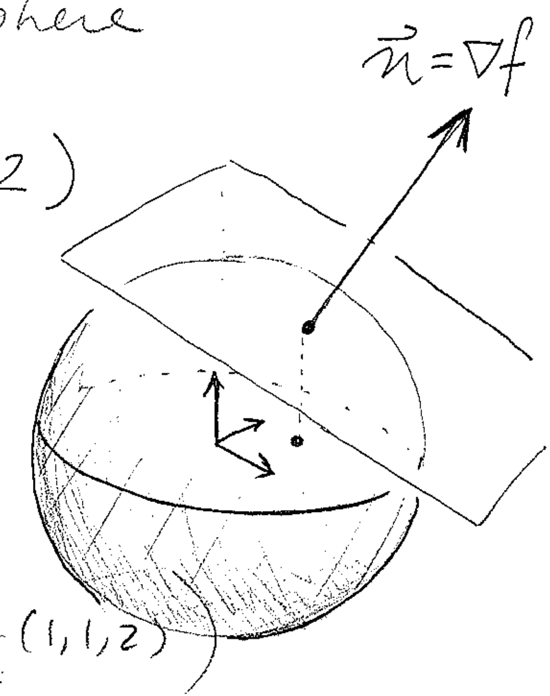
So $\nabla f(2,2) = (14, 14)$

and the line is: $x + y = 4$.

[Also works for more variables].

Q: Find the tangent to the sphere

$$x^2+y^2+z^2 = 6 \quad \text{at } (1,1,2)$$

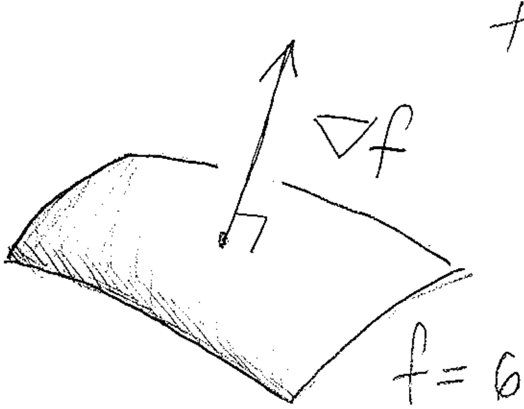


A. Take $f(x,y,z) = x^2+y^2+z^2$.

Then $\nabla f(1,1,2)$

$$= \left(\frac{\partial f}{\partial x}(1,1,2), \frac{\partial f}{\partial y}(1,1,2), \frac{\partial f}{\partial z}(1,1,2)\right)$$

$= (2, 2, 4)$ is at $\perp$ with the level set (= sphere) and so is normal to the tangent plane.

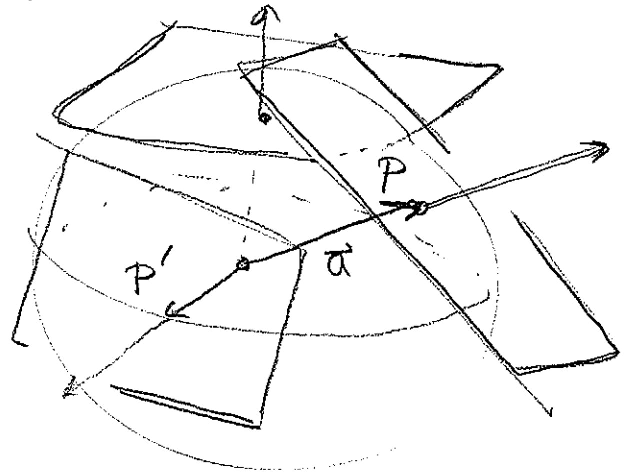


So

$$2\cdot(x-1)+2(y-1)+4\cdot(z-2)=0$$

$$\iff \quad x+y+2z=6$$

General Fact: If S is a sphere centered at O, and P a pt in S, then $\vec{a}=\overrightarrow{OP}$ is a normal vector to the tangent plane at P



Reason:

$$f(x,y,z)=x^2+y^2+z^2$$

$$\nabla f(x,y,z)=(2x,2y,2z)=2(x,y,z)$$

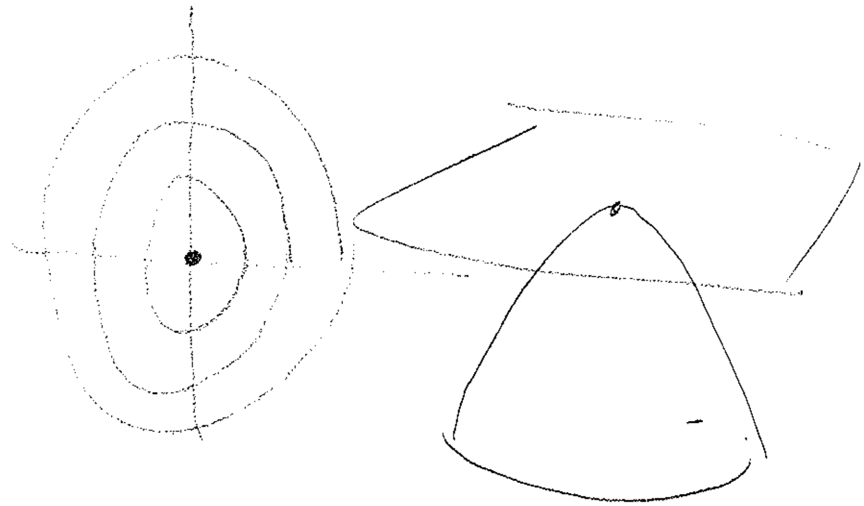
Back to $f:\mathbb{R}^2\to\mathbb{R}$.

Optimization: Finding min/max.

Suppose (a,b) is where $f(a,b)$ is largest. What is $\nabla f(a,b) = ?$

<u>A</u>. $\nabla f(a,b) = \vec{0}$.

[Can't increase f, afterall.]



Suppose instead $f(a,b)$ is the min value of f. What is $\nabla f(a,b) = ?$ <u>A</u>. $\vec{0}$.

How can we tell the two appart?

Need a 2^{nd} derivative test:

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \underbrace{\frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}}_{\text{Typically equal (Clairaut's Thm)}}$$

e.g. if both are coninnous.