Lecture 16: Constrained min/max. (14.8)

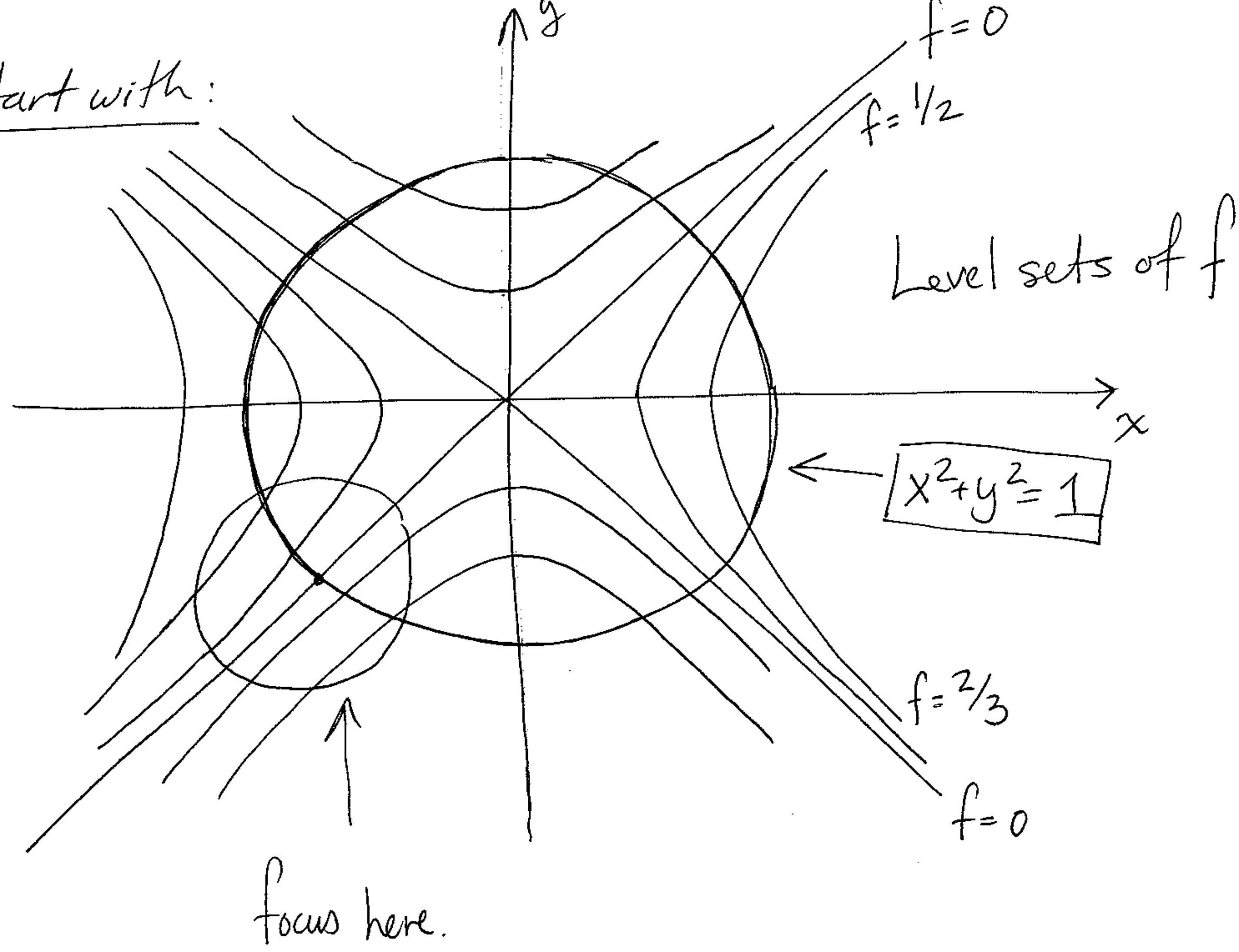
Last time:

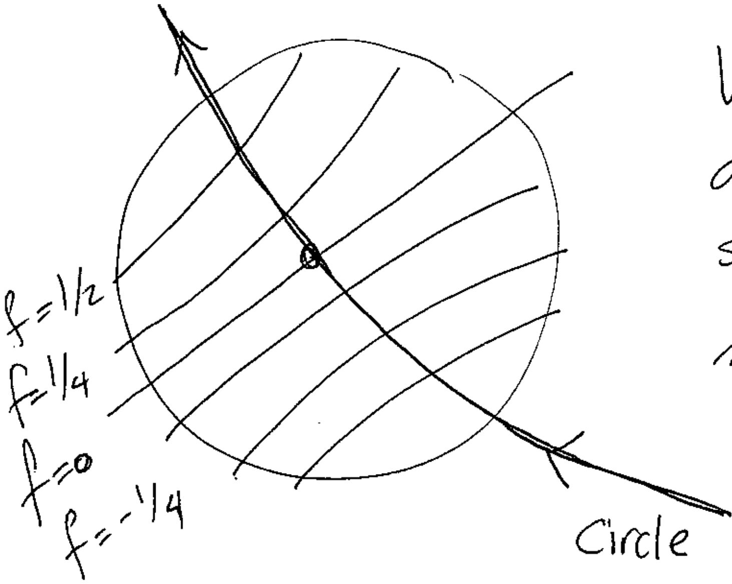
Extreme Value Thm: f continuous on D in $\mathbb{R}^n$. If D is closed and bounded, then f has a global min/max on D. These occur at a critical point of f or on the boundary of D.

focus today.

Ex: Find the max of $f(x,y) = x^2 - y^2$ on the unit circle.

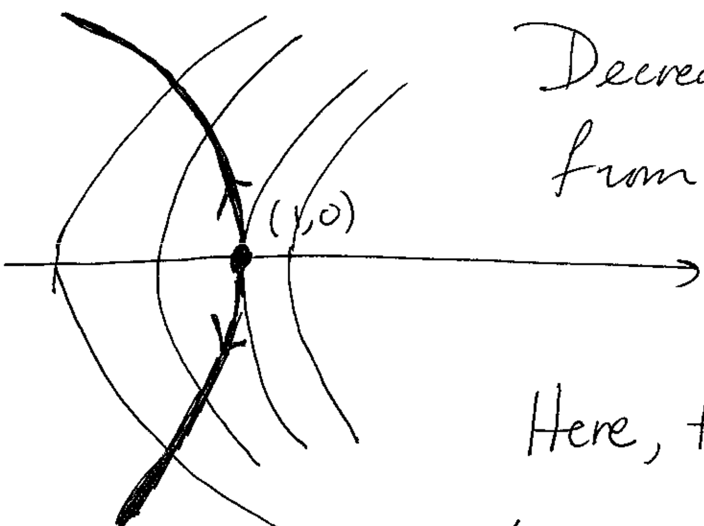
Start with:





When we have this picture, don't have a local ~~extremum~~ max since can increase f by moving clockwise along the circle.

However, when the level set of f is tangent to the circle, we can have a local max:



Decrease in f as we move away from $(1,0)$ along the circle in either direction.

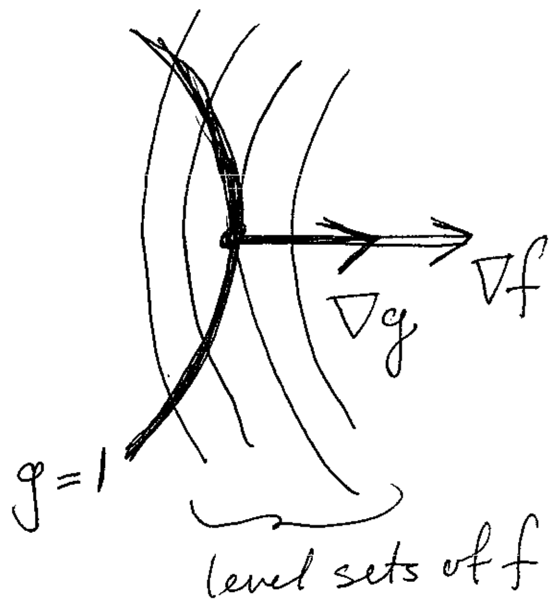
Here, there are four such tangencies:

	$(-1,0)$	$(1,0)$	$(0,1)$	$(0,-1)$
Value of f	1	1	-1	-1
	max	max	min	min

global in each case; the circle is closed and bounded.

Finding these tangencies in general:

View the circle as the level set $g(x,y)=1$ where $g(x,y)=x^2+y^2$. At a tangency, ∇f is at right angles to the circle:



Key: At a tangency, ∇f and ∇g point in the same direction:

$$\nabla f = \lambda \nabla g$$

λ ← some number.

$\boxed{\text{Lagrange Multipliers discovered by Euler}}$

Ex: $g(x,y)=x^2+y^2=1$

$$\nabla f = (2x,-2y) = \lambda \nabla g = \lambda(2x,2y) = (2\lambda x, 2\lambda y)$$

So $2x = 2\lambda x$ and $-2y = 2\lambda y$.

If $x \neq 0$, then $\lambda = 1$ and $y=0 \Rightarrow x = \pm 1$.

If $y \neq 0$, then $\lambda = -1$ and $x = 0 \Rightarrow y = \pm 1$

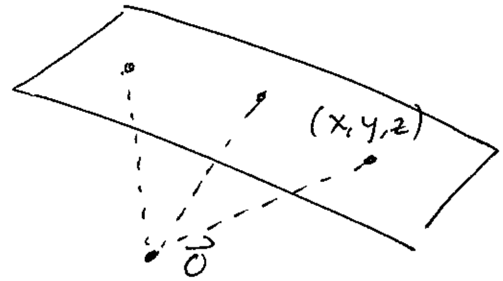
from $x^2+y^2=1$.

So the critical points are $(1,0)$ $(-1,0)$ $(0,1)$ $(0,-1)$ just as we found before.

<u>Ex</u>: Find the distance from $\overbrace{x-y+2z}^{g(x,y)} = 3$ to $\vec{0}$

<u>Minimize</u>: $f(x,y,z) = x^2+y^2+z^2$

<u>Subject to</u>: $g(x,y,z) = 3$



<u>Crit pts</u>: $\nabla f = \lambda \nabla g = (\lambda, -\lambda, 2\lambda)$

$\nabla f = (2x, 2y, 2z)$

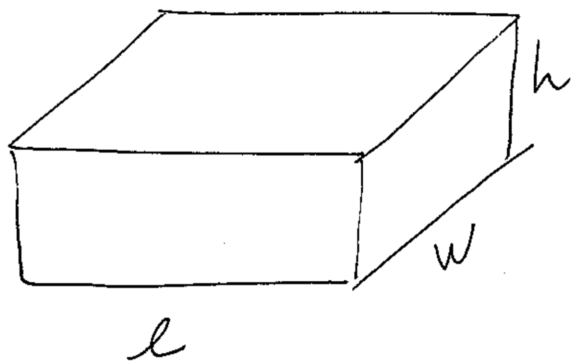
$\nabla g = (1, -1, 2)$

$$\begin{aligned} 2x &= \lambda \\ 2y &= -\lambda \\ 2z &= 2\lambda \end{aligned} \Rightarrow \begin{aligned} y &= -x \\ z &= 2x \end{aligned}$$

Combine with $g = 3$ gives $x - (-x) + 2(2x) = 3$ and so $x = 1/2$, and $y = -1/2$, $z = 1$ just as on Monday.

<u>Key Features</u>: ① Algebra easier than before.
② Don't have to solve for one var, which we won't be able to do for complicated g

Ex: Find the rectangular box of area 6 and largest volume. [What do you expect the ans to be?]



Maximize: $V = lwh$

Subject to: $A = 2lh + 2lw + 2wh = 6.$

$$\nabla V = (wh, lh, lw) = \lambda \nabla A = \lambda 2(h+w, l+h, l+w)$$

$$\Rightarrow \frac{1}{2\lambda} = \frac{1}{w} + \frac{1}{h} = \frac{1}{h} + \frac{1}{l} = \frac{1}{w} + \frac{1}{l}$$

Q: Why does this crit pt have to be a max?

$$\Rightarrow \frac{1}{l} = \frac{1}{w} = \frac{1}{h} \Rightarrow l = w = h$$

Combine with $A = 6$ gives $6l^2 = 6 \Rightarrow l = w = h = 1.$

Point: V and A is very symmetric. In particular if (l_0, w_0, h_0) is a crit pt, so is (w_0, h_0, l_0) and (w_0, l_0, h_0), ... Hence if there is only one crit pt, it must be symmetric.