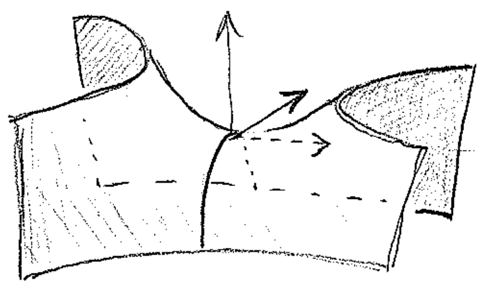
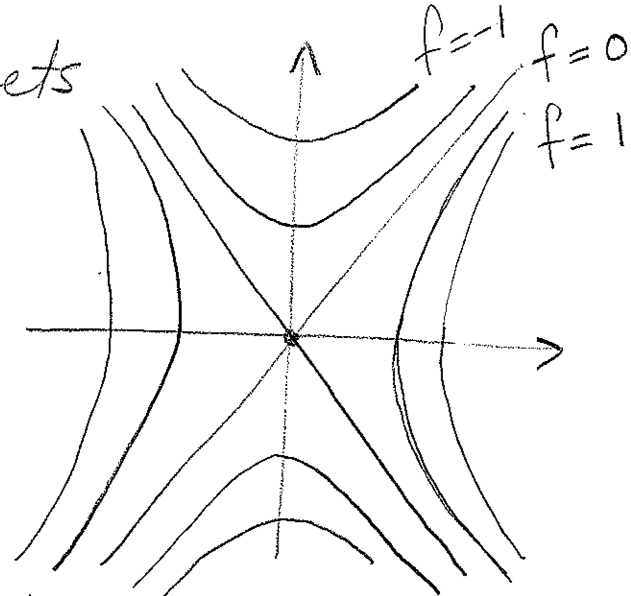
Lecture 6: Level sets in 3^d (14.1), quadric surfaces (12.6), review of limits (14.2)

Last time: $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x^2 - y^2$

Graph



Level sets



For $f: \mathbb{R}^3 \to \mathbb{R}$ can't draw the graph (in $\mathbb{R}^4$) but can still look at level sets.

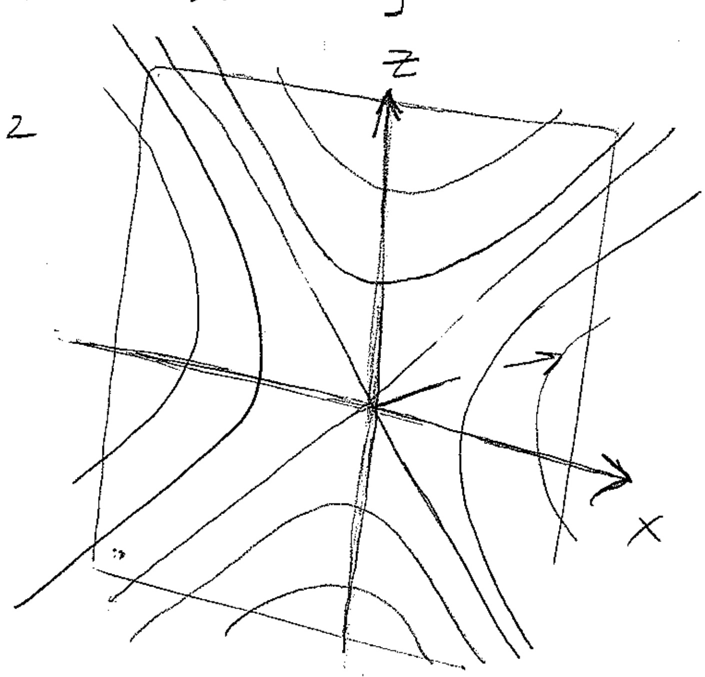
[Did $f(x,y,z) = x^2+y^2+z^2$ last time]

Ex: $f(x,y,z) = x^2+y^2-z^2$

First, look at xz-plane

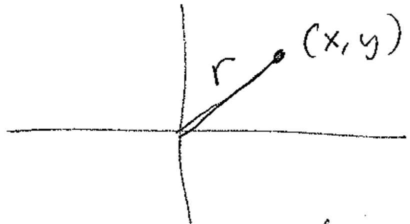
$$f(x,0,z) = x^2 - z^2$$

so the level sets

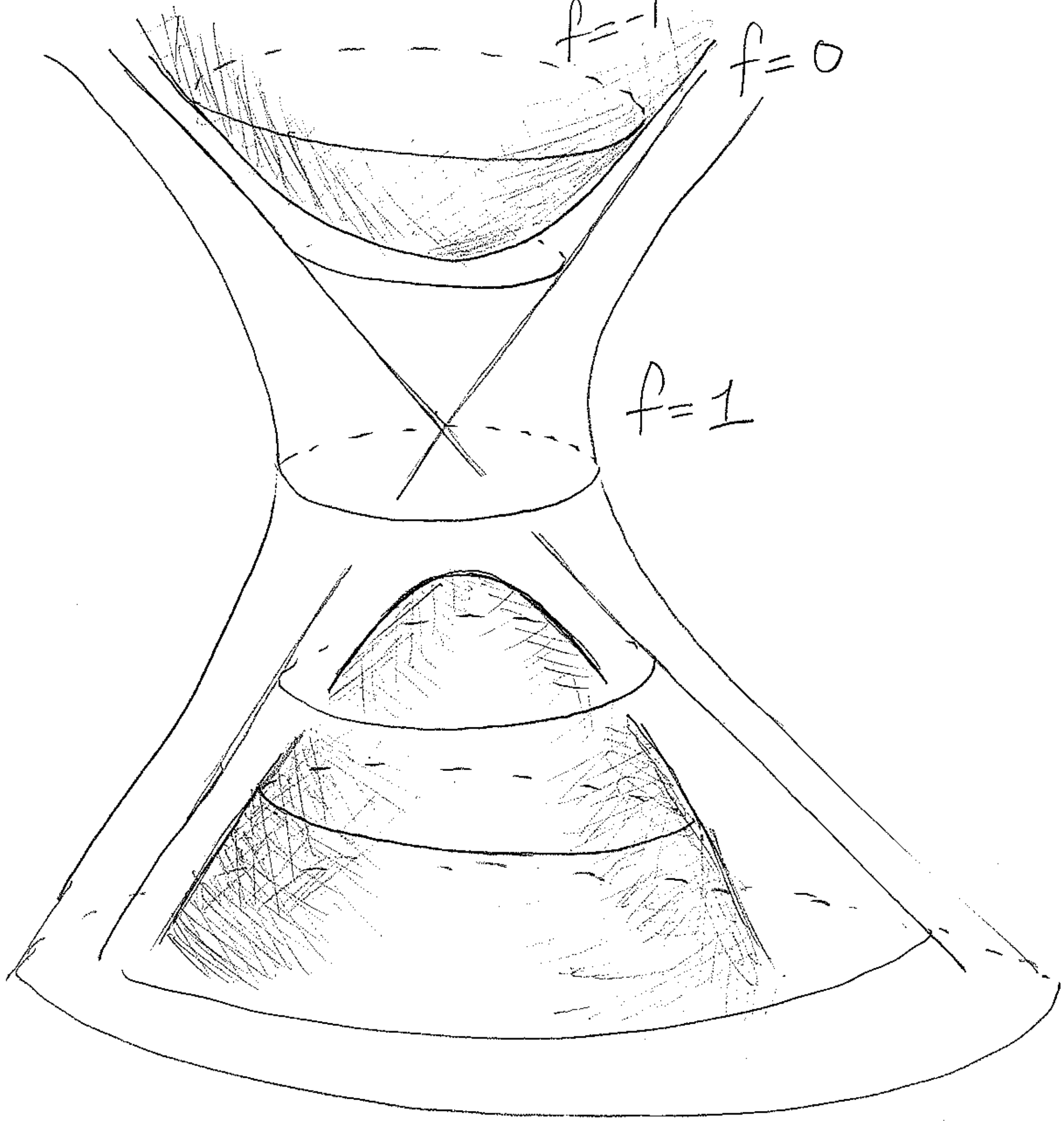


in this plane look like what we had on Wed.

Also, as $x^2+y^2=r^2$



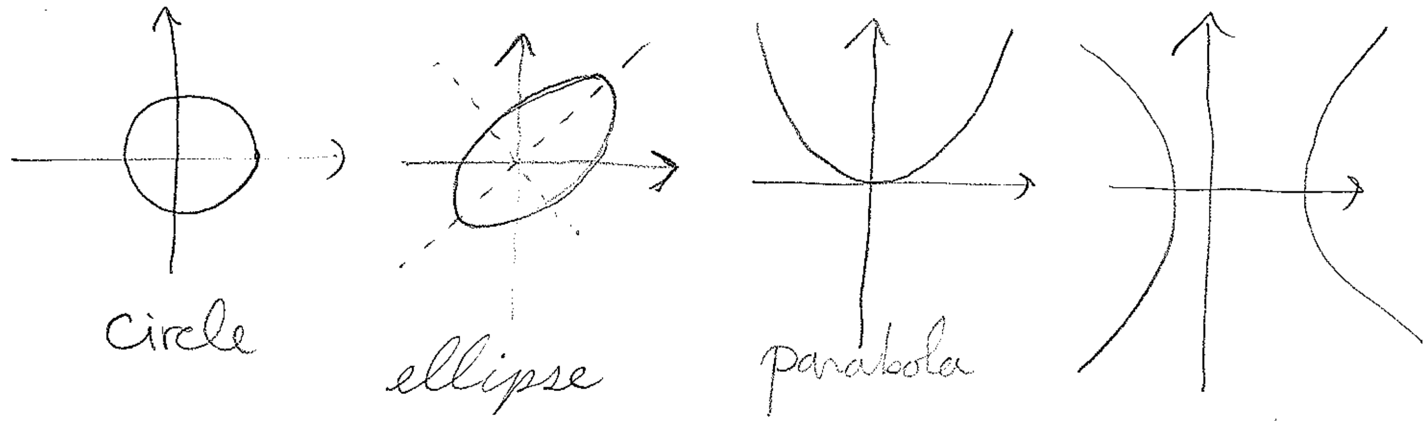
we have $f(x,y,z) = r^2 - z^2$ and so each level set is symmetric about z-axis:



[These level sets are all examples of quadric surfaces.]

Conic Sections: Solutions in $\mathbb{R}^2$

of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

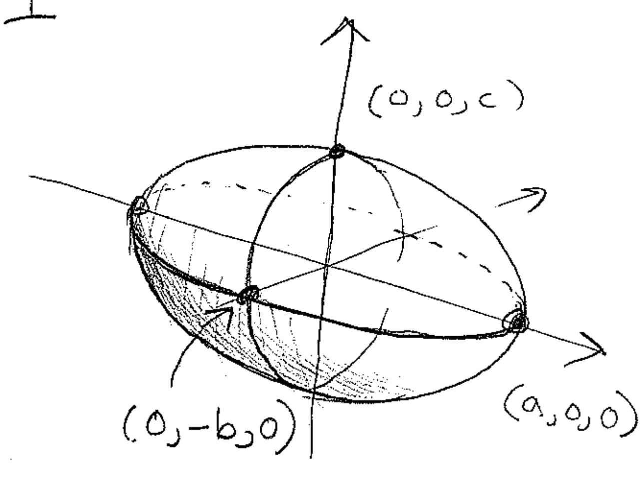


hyperbola

Quadric Surfaces in $\mathbb{R}^3$ (Section 12.6)

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

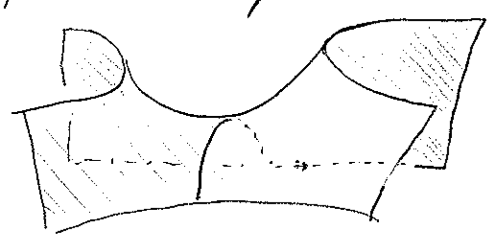
Ex: Ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$



Elliptic parabolid:

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

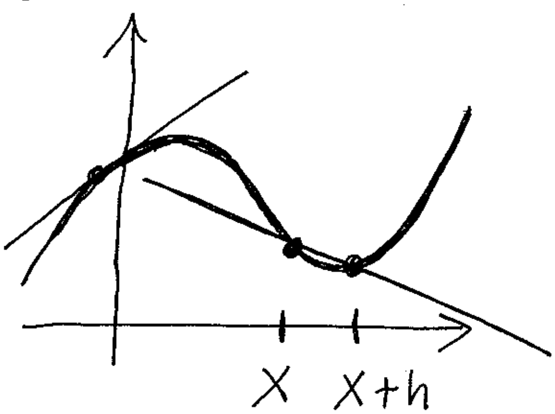
Hyperbolic parabolid:



$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

The other quadric surfaces are the double cone and the hyperboloid (of 1 and 2 sheets) that we saw at the start.

<u>Limits</u> (14.2) [To talk about derivatives, first need to discuss limits for fns of several vars]



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

[Can take different perspectives on limits — focus on them as a way of estimating error.]

What does a calculator do to compute

$$\sin(2) = 0.9092974268$$

Uses

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

But how many terms do we need to add up?

Consider $E : \mathbb{R} \to \mathbb{R}$ (an "error function")

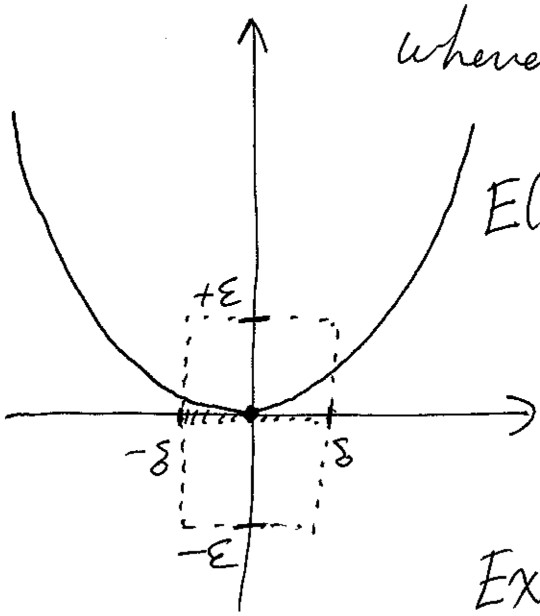
We say

target error bound

$$\lim_{h \to 0} E(h) = 0 \text{ if given } \boxed{\varepsilon > 0} \text{ we can}$$

always find $\delta > 0$ so that

whenever $|h| < \delta$ then $|E(h)| < \varepsilon$.



$E(h) = h^2$ ——— ◦ ———

View as challenge-response process.

<u>Ex</u>: $E(h) = h^2 \quad \varepsilon = 1/10$

Take $\delta = 1/4$. If $|h| < \delta = 1/4$, then

$$|E(h)| = |h^2| = |h|^2 < \frac{1}{16} < \frac{1}{10}$$

<u>Ex</u>: $\varepsilon = 1/100 \quad \delta =$ Audience response

$\varepsilon = 1/1000 \quad \delta =$ — " —

Claim: $\lim\limits_{h \to 0} h^2 = 0$

Reason: If you give me $\varepsilon > 0$, I'll take $\delta = \sqrt{\varepsilon}$. Then if $|h| < \delta$ we have

$$|h^2| = |h|^2 < \delta^2 = \varepsilon. \; \boxed{\checkmark}$$

Ex: $E(h) = 2h + h^2$ Know $\lim\limits_{h \to 0} 2h + h^2 = 0$

Given $\varepsilon = 1/10$ take $\delta = \frac{1}{100}$.

If $|h| < \delta$, then
$$|2h + h^2| \le 2|h| + |h|^2 < 2 \cdot \frac{1}{100} + \frac{1}{10{,}000} < \frac{3}{100} < 1/10 = \varepsilon.$$

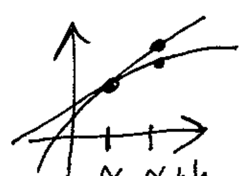
In general, say

$$\lim_{x \to a} f(x) = c$$

if $f(a+h) = c + E(h)$

where $\lim\limits_{h \to 0} E(h) = 0$.

Differentiability:



$$f(x+h) = f(x) + f'(x)h + E(h)$$

where $E(h)$ is <u>really</u> small, i.e.

$$\lim_{h \to 0} \frac{E(h)}{h} = 0.$$