

Math 418: Problem Set 1.¹

Due date: In class on Wednesday, January 27.

Webpage: <http://dunfield.info/418>

Office hours: Monday 10-11, Tuesday 2-4 (shifted one hour earlier than normal).

Textbook: Dummit and Foote, *Abstract Algebra*, 3rd edition.

1. The background on rings I will be assuming is the contents of Chapter 7 of our text. Look through that chapter and find one thing you don't understand and work a problem from the text about it. (If everything in the chapter is clear, just pick some problem that's amusing and do that. If no problem looks amusing, you're in the wrong class.)
2. Consider $R = \mathbb{Z}[\sqrt{-5}]$ with the norm $N: R \rightarrow \mathbb{Z}_{\geq 0}$ given by $N(a) = |a|^2$. (This isn't a norm in the sense of Euclidean domains, but rather another sense we'll learn about later.) In the below problems, a key is that $N(a \cdot b) = N(a)N(b)$.
 - (a) Prove that $a \in R$ is a unit if and only if $N(a) = 1$. Find all the units in R .
 - (b) Recall that $r \in R$ is irreducible if whenever $r = ab$ then one of a or b is a unit. Use the norm to show that $2, 3, 1 + \sqrt{-5}$, and $1 - \sqrt{-5}$ are all irreducible elements of R .
 - (c) Show that $2, 3, 1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are not unit multiples of one another. Thus R lacks unique factorization since $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. Weird, huh?
3. Let R be an integral domain. Recall that g is a greatest common divisor of two elements a, b if g divides a and b , and if d is any other common divisor then $d|g$.
 - (a) If g and g' are two gcds of elements (a, b) show that $g' = ug$ for some unit u .
 - (b) Again consider the ring $R = \mathbb{Z}[\sqrt{-5}]$. Prove that 6 and $2 + 2\sqrt{-5}$ have no gcd. Hint: Use the common divisors 2 and $1 + \sqrt{-5}$.
4. Let R be a Principal Ideal Domain, and I an ideal of R . Prove that every ideal of $S = R/I$ is principal. (S may fail to be an integral domain, and hence not always a P.I.D.)

¹Revised version of Jan. 25 correcting an error in Problem 4.