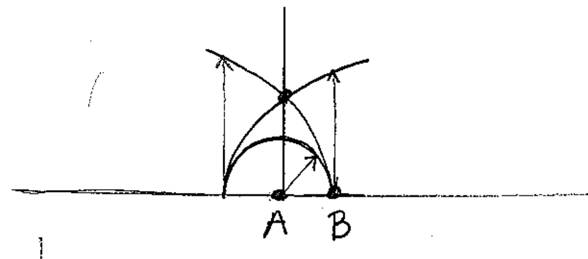
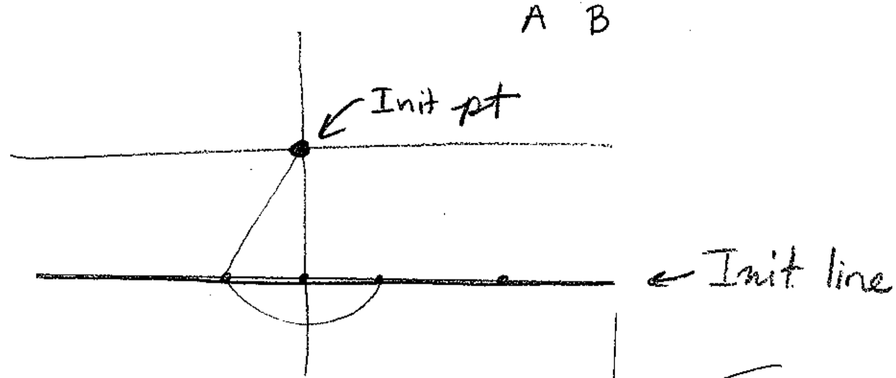
Lecture 12: Limitations of Straightedge and Compass.

Given a ruler and compass, what can you construct?

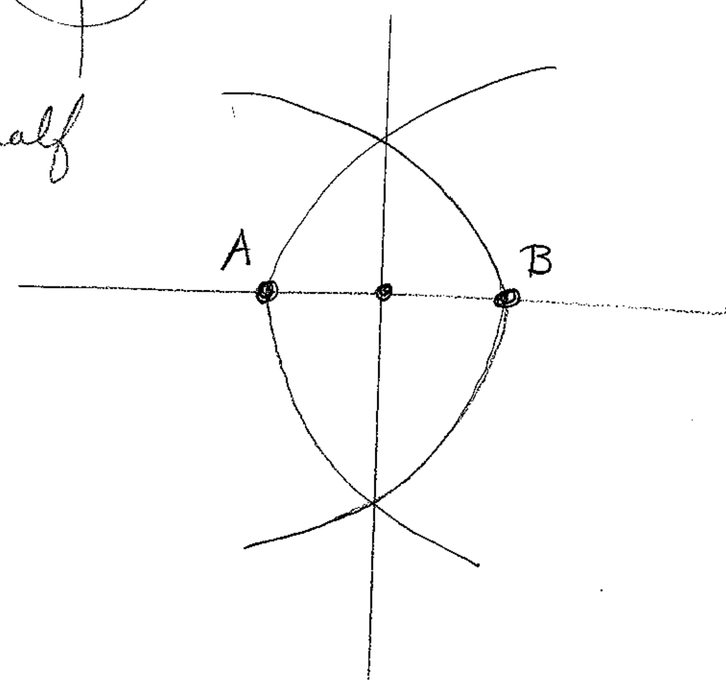
(2) Perpendicular Bisectors:



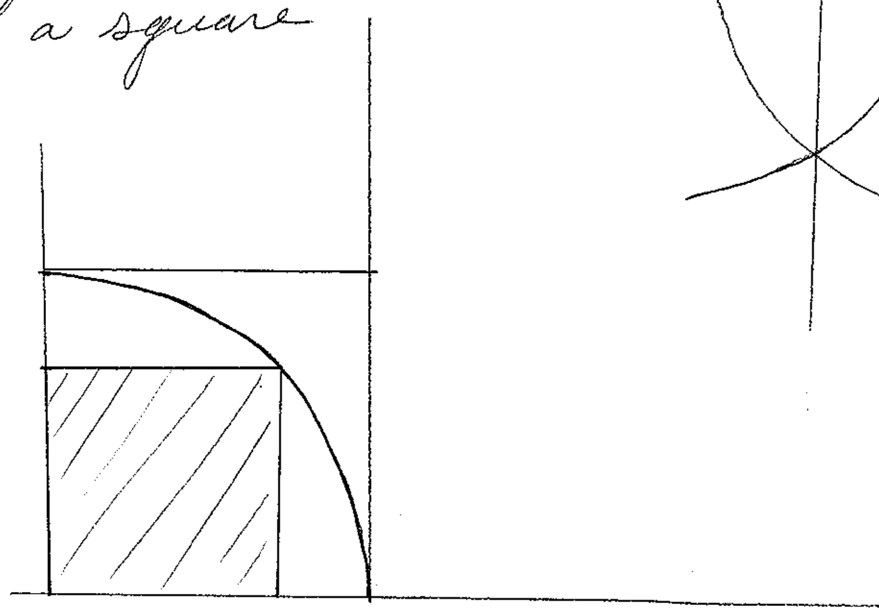
(3) Parallel lines:



(1) Divide a segment in half



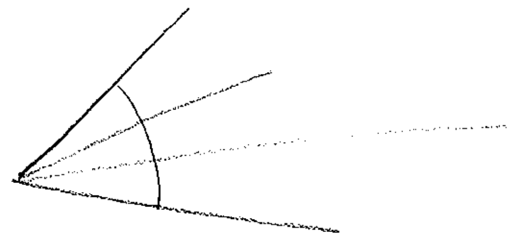
(4) Double the area of a square



(5) Make a regular 17-gon. (Or a 65537-gon!)

Things you can't do

(1) Trisect an angle



(2) Given a circle, build a square of the same area.

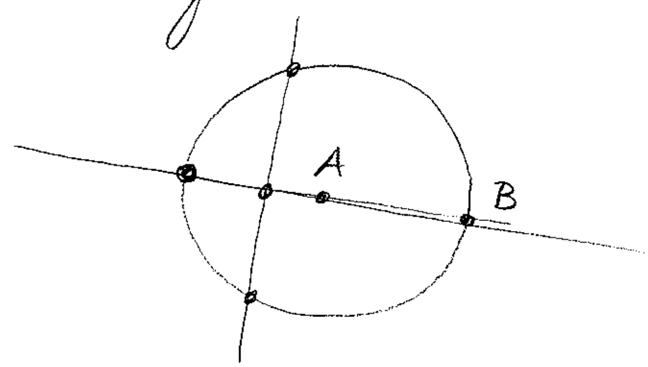
<u>Rules</u>: (Goes back to the ancient Greeks)

(A) Given two points can draw

 (1) The line joining them

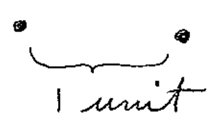
 (2) The circle centered at one pt, passing through the other.

(B) Find the points of intersection of lines and circles



<u>Note</u>: Can't measure things.

<u>Starting Setup</u>: Two points:



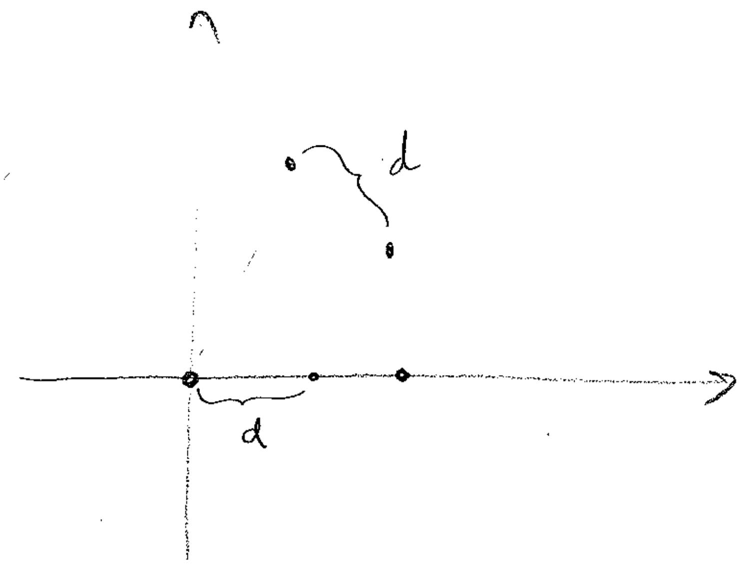
Consider the constructable numbers:

$$C = \left\{ d \in \mathbb{R} \;\middle|\; \begin{array}{l} \text{starting with can construct} \\ \text{two points } x, y \text{ with } d(x,y) = \pm d \end{array} \right\}$$

Observations:

① Impose a Cartesian coordinate system with one pt at 0 and the other at $(1,0)$

Prop: $d \in C$ iff can construct the point $(d,0)$.

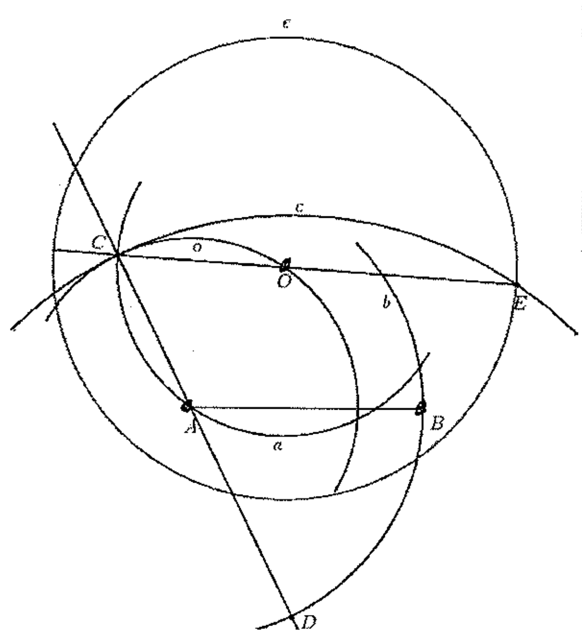


Pf: $(\Leftarrow)$ clear

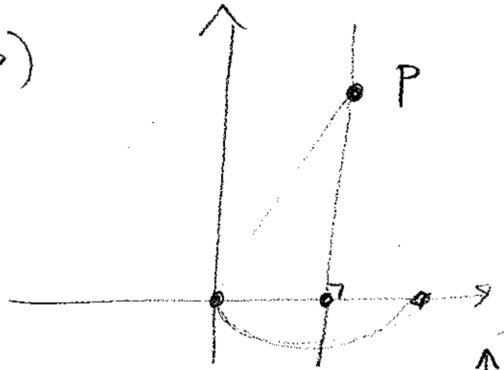
$(\Rightarrow)$ Given 3 points, can construct a circle centered about one of radius the dist between the others

[Or use a modern compass.]

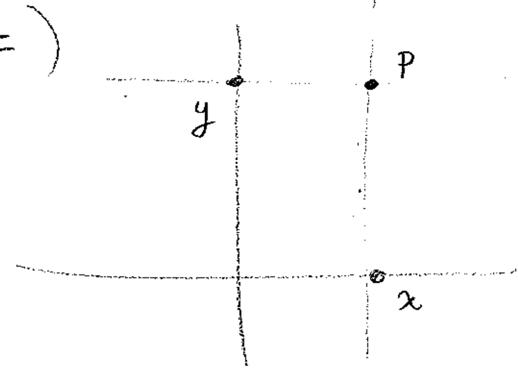
Prop: A point $p \in \mathbb{R}^2$ is constructible $\Longleftrightarrow$ both coords are in C.



Pf: ($\Rightarrow$)

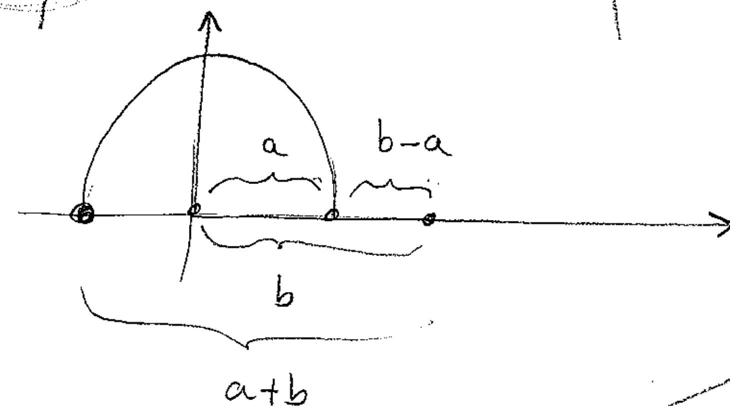


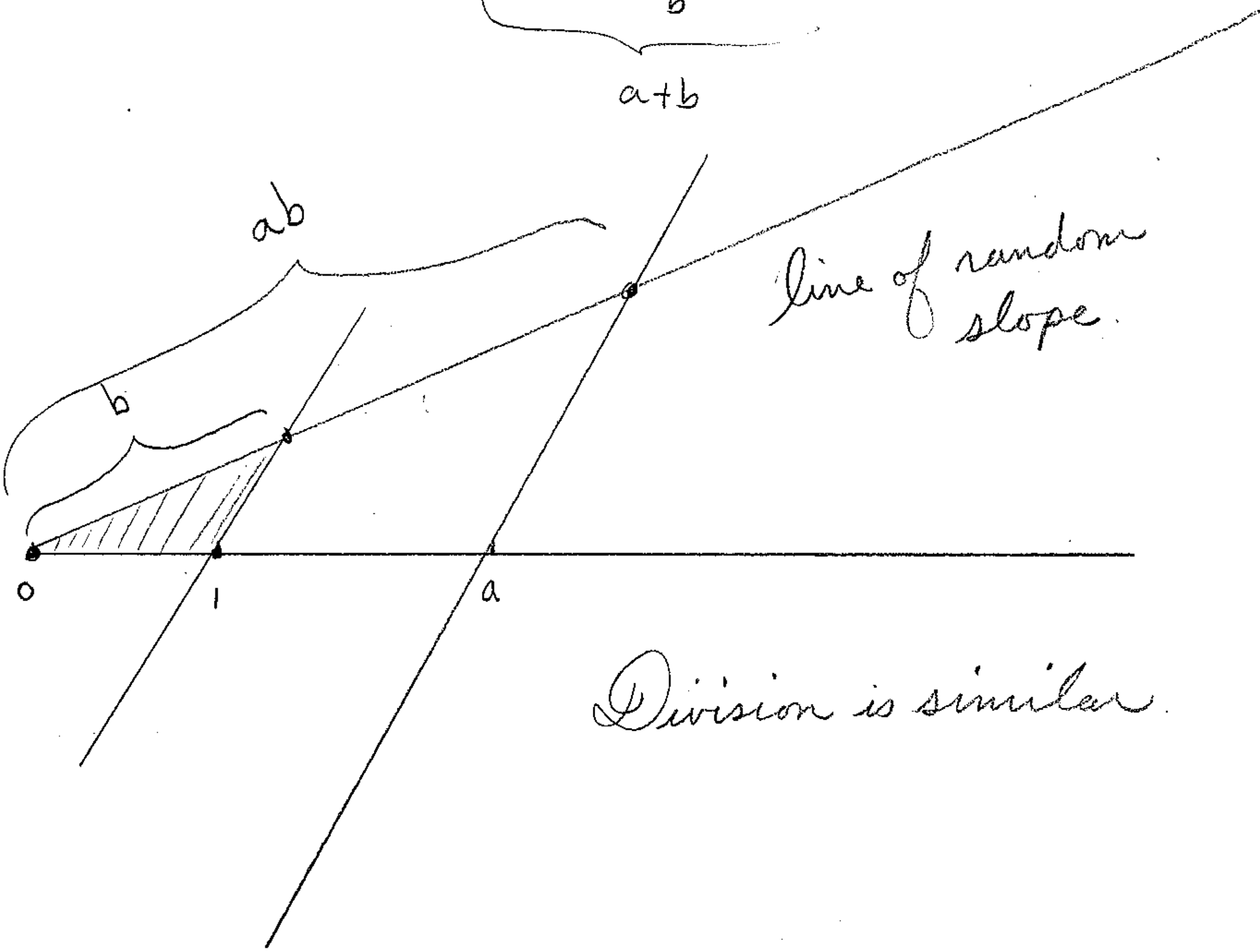
($\Leftarrow$)



∎

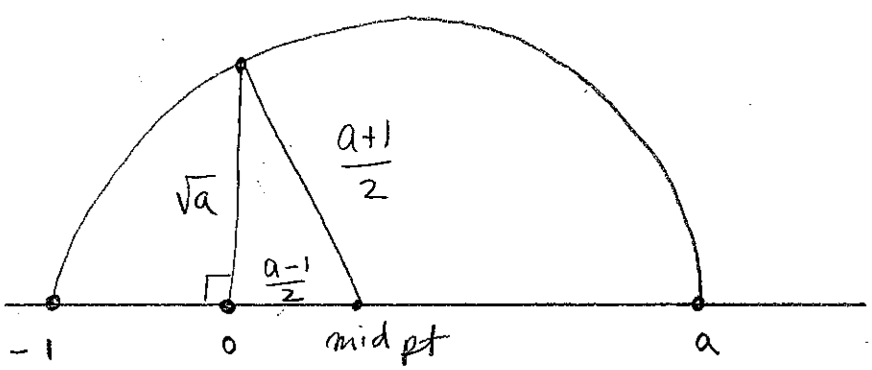
② C is field.





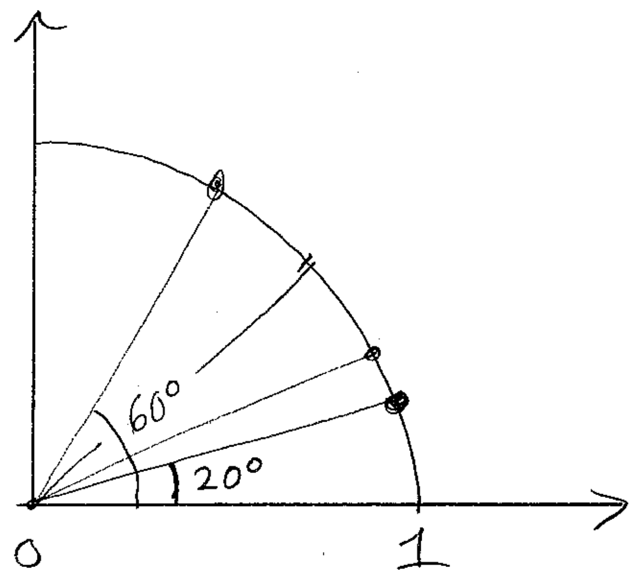
Division is similar.

③ If $a \in C$, so is $\sqrt{a}$



<u>Thm:</u> $a \in C$. Then $[\mathbb{Q}(a) : \mathbb{Q}] = 2^n$

<u>Cor:</u> Can't trisect angles



Pf: If so, then can construct the pt on the unit circle with angle $20° = \pi/9$.

Thus $c = \cos \pi/9$ is in C. By the tripple angle formula

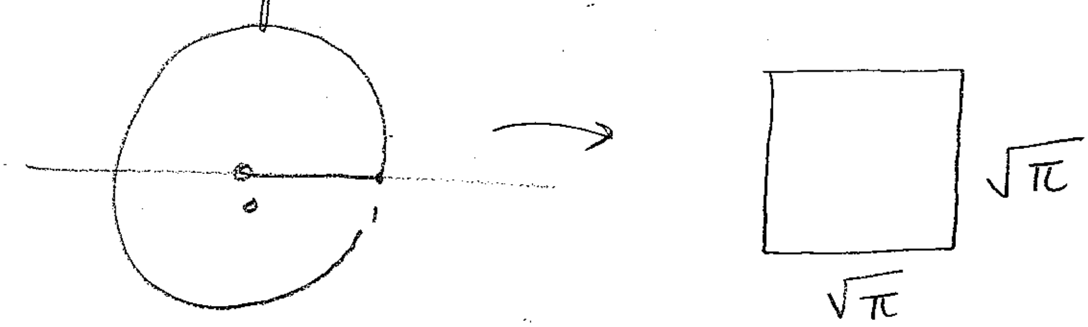
$$1/2 = \cos \pi/3 = 4\cos^3 \pi/9 - 3\cos \pi/9$$

$$\Rightarrow \ 8c^3 - 6c - 1 = 0$$

Since $8x^3 - 6x - 1$ is irred in $\mathbb{Z}[x]$ (and hence $\mathbb{Q}[x]$) (reduce mod 3), $[\mathbb{Q}(c) : \mathbb{Q}] = 3$. But then c can't be in C.

Cor: Can't square the circle

Pf:



Area $= \pi$

Would imply that $\sqrt{\pi} \in C \Rightarrow \pi \in C$

$\Rightarrow [\mathbb{Q}(\pi) : \mathbb{Q}] < \infty$, but π is transcendental. ∎

Proof of Thm: Fix $a \in C$. Then $(a, 0)$ is const in a series of steps. Let $F_n = \mathbb{Q}(\text{coor of first } n \text{ points constructed})$

Then $F_{n+1} = F_n(\underbrace{a_{n+1}, b_{n+1}}_{\text{coor of } P_{n+1}})$

Now P_{n+1} is found by intersecting lines and circles, and the eqns for such have coeffs in $\underline{F_n}$.

Intersecting two lines:

linear equations $\Rightarrow F_{n+1} = F_n$

Intersecting a line with a circle: Degree 2 extension

Intersecting two circles: reduces to previous case

Details next time.