2. To add two vectors geometrically, we can use either the Triangle Law or the Parallelogram Law, as illustrated in Figures 3 and 4 in Section 12.2. Algebraically, we add the corresponding components of the vectors.

4. See (1) in Section 12.2.

6. The dot product can be used to find the angle between two vectors and the scalar projection of one vector onto another. In particular, the dot product can determine if two vectors are orthogonal. Also, the dot product can be used to determine the work done moving an object given the force and displacement vectors.

8. See Theorem 12.4.9 and the preceding discussion; use either (4) or (7) in Section 12.4.

10. (a) The area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$ is the length of the cross product: $|\mathbf{a} \times \mathbf{b}|$.

(b) The volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product: $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

12. The angle between two intersecting planes is defined as the acute angle between their normal vectors. We can find this angle using Corollary 12.3.6.

14. See (5), (6), and (7) in Section 12.5.

16. (a) Determine the vectors $\overrightarrow{PQ} = \langle a_1, a_2, a_3 \rangle$ and $\overrightarrow{PR} = \langle b_1, b_2, b_3 \rangle$. If there is a scalar t such that $\langle a_1, a_2, a_3 \rangle = t \langle b_1, b_2, b_3 \rangle$, then the vectors are parallel and the points must all lie on the same line.

Alternatively, if $\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{0}$, then $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel, so P, Q, and R are collinear.

Thirdly, an algebraic method is to determine an equation of the line joining two of the points, and then check whether or not the third point satisfies this equation.

(b) Find the vectors $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{PR} = \mathbf{b}$, $\overrightarrow{PS} = \mathbf{c}$. $\mathbf{a} \times \mathbf{b}$ is normal to the plane formed by P, Q and R, and so S lies on this plane if $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$ are orthogonal, that is, if $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$. (Or use the reasoning in Example 5 in Section 12.4.) Alternatively, find an equation for the plane determined by three of the points and check whether or not the fourth point satisfies this equation.

18. The traces of a surface are the curves of intersection of the surface with planes parallel to the coordinate planes. We can find the trace in the plane $x = k$ (parallel to the yz-plane) by setting $x = k$ and determining the curve represented by the resulting equation. Traces in the planes $y = k$ (parallel to the xz-plane) and $z = k$ (parallel to the xy-plane) are found similarly.

1. This is false, as the dot product of two vectors is a scalar, not a vector.

2. False. For example, if $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = -\mathbf{i}$ then $|\mathbf{u} + \mathbf{v}| = |\mathbf{0}| = 0$ but $|\mathbf{u}| + |\mathbf{v}| = 1 + 1 = 2$.

3. False. For example, if $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$ then $|\mathbf{u} \cdot \mathbf{v}| = |0| = 0$ but $|\mathbf{u}|\,|\mathbf{v}| = 1 \cdot 1 = 1$. In fact, by Theorem 12.3.3, $|\mathbf{u} \cdot \mathbf{v}| = \big||\mathbf{u}|\,|\mathbf{v}| \cos\theta\big|$.

4. False. For example, $|\mathbf{i} \times \mathbf{i}| = |\mathbf{0}| = 0$ (see Example 12.4.2) but $|\mathbf{i}|\,|\mathbf{i}| = 1 \cdot 1 = 1$. In fact, by Thereom 12.4.9, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$.

5. True, by Theorem 12.3.2, property 2.

6. False. Property 1 of Theorem 12.4.11 says that $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

7. True. If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then by Theorem 12.4.9, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta = |\mathbf{v}|\,|\mathbf{u}| \sin\theta = |\mathbf{v} \times \mathbf{u}|$. (Or, by Theorem 12.4.11, $|\mathbf{u} \times \mathbf{v}| = |-\mathbf{v} \times \mathbf{u}| = |-1|\,|\mathbf{v} \times \mathbf{u}| = |\mathbf{v} \times \mathbf{u}|$.)

8. This is true by Theorem 12.3.2, property 4.

9. Theorem 12.4.11, property 2 tells us that this is true.

10. This is true by Theorem 12.4.11, property 4.

11. This is true by Theorem 12.4.11, property 5.

12. In general, this assertion is false; a counterexample is $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$. (See the paragraph preceding Theorem 12.4.11.)

13. This is true because $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ (see Theorem 12.4.8), and the dot product of two orthogonal vectors is $\mathbf{0}$.

14.
$$\begin{aligned}(\mathbf{u}+\mathbf{v}) \times \mathbf{v} &= \mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{v} && \text{[by Theorem 12.4.11, property 4]}\\ &= \mathbf{u} \times \mathbf{v} + \mathbf{0} && \text{[by Example 12.4.2]}\\ &= \mathbf{u} \times \mathbf{v}, \text{ so this is true.}\end{aligned}$$

15. This is false. A normal vector to the plane is $\mathbf{n} = \langle 6, -2, 4 \rangle$. Because $\langle 3, -1, 2 \rangle = \frac{1}{2}\mathbf{n}$, the vector is parallel to $\mathbf{n}$ and hence perpendicular to the plane.

16. This is false, because according to Equation 12.5.8, $ax + by + cz + d = 0$ is the general equation of a plane.

17. This is false. In $\mathbb{R}^2$, $x^2 + y^2 = 1$ represents a circle, but $\{(x, y, z) \mid x^2 + y^2 = 1\}$ represents a *three-dimensional surface*, namely, a circular cylinder with axis the z-axis.

18. This is false. In $\mathbb{R}^3$ the graph of $y = x^2$ is a parabolic cylinder (see Example 12.6.1). A paraboloid has an equation such as $z = x^2 + y^2$.

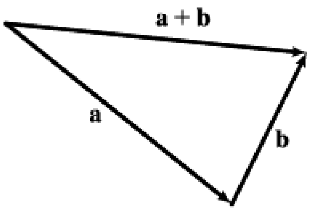
19. False. For example, $\mathbf{i}\cdot\mathbf{j}=\mathbf{0}$ but $\mathbf{i}\neq\mathbf{0}$ and $\mathbf{j}\neq\mathbf{0}$.

20. This is false. By Corollary 12.4.10, $\mathbf{u}\times\mathbf{v}=\mathbf{0}$ for any nonzero parallel vectors $\mathbf{u}$, $\mathbf{v}$. For instance, $\mathbf{i}\times\mathbf{i}=\mathbf{0}$.

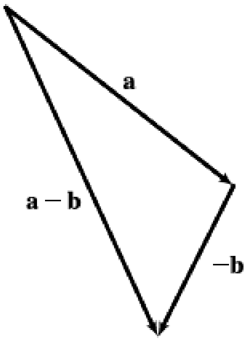
21. This is true. If $\mathbf{u}$ and $\mathbf{v}$ are both nonzero, then by (7) in Section 12.3, $\mathbf{u}\cdot\mathbf{v}=0$ implies that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. But $\mathbf{u}\times\mathbf{v}=\mathbf{0}$ implies that $\mathbf{u}$ and $\mathbf{v}$ are parallel (see Corollary 12.4.10). Two nonzero vectors can't be both parallel and orthogonal, so at least one of $\mathbf{u}$, $\mathbf{v}$ must be $\mathbf{0}$.

22. This is true. We know $\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}|\ |\mathbf{v}|\ \cos\theta$ where $|\mathbf{u}|\geq 0$, $|\mathbf{v}|\geq 0$, and $|\cos\theta|\leq 1$, so $|\mathbf{u}\cdot\mathbf{v}|=|\mathbf{u}|\ |\mathbf{v}|\ |\cos\theta|\leq|\mathbf{u}|\ |\mathbf{v}|$.

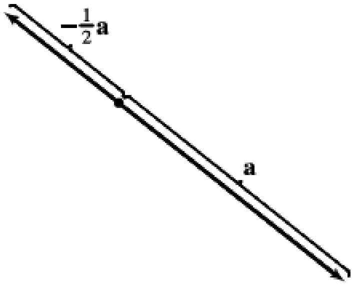
2. (a)



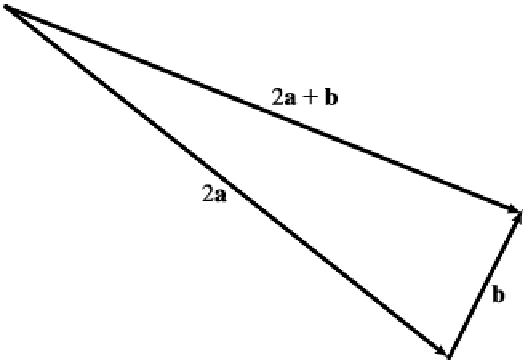
(b)



(c)



(d)



4. (a) $2\,\mathbf{a}+3\,\mathbf{b}=2\,\mathbf{i}+2\,\mathbf{j}-4\,\mathbf{k}+9\,\mathbf{i}-6\,\mathbf{j}+3\,\mathbf{k}=11\,\mathbf{i}-4\,\mathbf{j}-\mathbf{k}$

(b) $|\mathbf{b}|=\sqrt{9+4+1}=\sqrt{14}$

(c) $\mathbf{a}\cdot\mathbf{b}=(1)(3)+(1)(-2)+(-2)(1)=-1$

(d) $$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&-2\\3&-2&1\end{vmatrix}=(1-4)\,\mathbf{i}-(1+6)\,\mathbf{j}+(-2-3)\,\mathbf{k}=-3\,\mathbf{i}-7\,\mathbf{j}-5\,\mathbf{k}$$

(e) $$\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&-2&1\\0&1&-5\end{vmatrix}=9\,\mathbf{i}+15\,\mathbf{j}+3\,\mathbf{k},\quad |\mathbf{b}\times\mathbf{c}|=3\,\sqrt{9+25+1}=3\,\sqrt{35}$$

(f) $$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}1&1&-2\\3&-2&1\\0&1&-5\end{vmatrix}=\begin{vmatrix}-2&1\\1&-5\end{vmatrix}-\begin{vmatrix}3&1\\0&-5\end{vmatrix}-2\begin{vmatrix}3&-2\\0&1\end{vmatrix}=9+15-6=18$$

(g) $\mathbf{c}\times\mathbf{c}=\mathbf{0}$ for any $\mathbf{c}$.

(h) From part (e),

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{a}\times(9\,\mathbf{i}+15\,\mathbf{j}+3\,\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&-2\\9&15&3\end{vmatrix}$$

$$=(3+30)\,\mathbf{i}-(3+18)\,\mathbf{j}+(15-9)\,\mathbf{k}=33\,\mathbf{i}-21\,\mathbf{j}+6\,\mathbf{k}$$

(i) The scalar projection is $\operatorname{comp}_{\mathbf{a}}\mathbf{b}=|\mathbf{b}|\cos\theta=\mathbf{a}\cdot\mathbf{b}/\,|\mathbf{a}|=-\frac{1}{\sqrt{6}}$.

(j) The vector projection is $\operatorname{proj}_{\mathbf{a}}\mathbf{b}=-\dfrac{1}{\sqrt{6}}\left(\dfrac{\mathbf{a}}{|\mathbf{a}|}\right)=-\frac{1}{6}(\mathbf{i}+\mathbf{j}-2\,\mathbf{k})$.

(k) $\cos\theta=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}=\dfrac{-1}{\sqrt{6}\,\sqrt{14}}=\dfrac{-1}{2\,\sqrt{21}}$ and $\theta=\cos^{-1}\left(\dfrac{-1}{2\,\sqrt{21}}\right)\approx 96^\circ$.

6. We know that the cross product of two vectors is orthogonal to both. So we calculate

$(\mathbf{j}+2\,\mathbf{k})\times(\mathbf{i}-2\,\mathbf{j}+3\,\mathbf{k})=[3-(-4)]\,\mathbf{i}-(0-2)\,\mathbf{j}+(0-1)\,\mathbf{k}=7\,\mathbf{i}+2\,\mathbf{j}-\mathbf{k}$.

Then two unit vectors orthogonal to both given vectors are $\pm\dfrac{7\,\mathbf{i}+2\,\mathbf{j}-\mathbf{k}}{\sqrt{7^2+2^2+(-1)^2}}=\pm\dfrac{1}{3\,\sqrt{6}}\,(7\,\mathbf{i}+2\,\mathbf{j}-\mathbf{k})$,

that is, $\frac{7}{3\,\sqrt{6}}\,\mathbf{i}+\frac{2}{3\,\sqrt{6}}\,\mathbf{j}-\frac{1}{3\,\sqrt{6}}\,\mathbf{k}$ and $-\frac{7}{3\,\sqrt{6}}\,\mathbf{i}-\frac{2}{3\,\sqrt{6}}\,\mathbf{j}+\frac{1}{3\,\sqrt{6}}\,\mathbf{k}$.

8. $(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] = (\mathbf{a} \times \mathbf{b}) \cdot ([(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}]\,\mathbf{c} - [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{c}]\,\mathbf{a})$

[by Property 6 of Theorem 12.4.11]

$$= (\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}]\,\mathbf{c} = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]\,(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

$$= [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]\,[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})] = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]^2$$

10. $\overrightarrow{AB} = \langle 1, 3, -1 \rangle$, $\overrightarrow{AC} = \langle -2, 1, 3 \rangle$ and $\overrightarrow{AD} = \langle -1, 3, 1 \rangle$. By Equation 12.4.13,

$$\overrightarrow{AB} \cdot \left(\overrightarrow{AC} \times \overrightarrow{AD}\right) = \begin{vmatrix} 1 & 3 & -1 \\ -2 & 1 & 3 \\ -1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} - 3 \begin{vmatrix} -2 & 3 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 1 \\ -1 & 3 \end{vmatrix} = -8 - 3 + 5 = -6.$$

The volume is $\left|\overrightarrow{AB} \cdot \left(\overrightarrow{AC} \times \overrightarrow{AD}\right)\right| = 6$ cubic units.

12. $\mathbf{D} = 4\,\mathbf{i} + 3\,\mathbf{j} + 6\,\mathbf{k}$, $\quad W = \mathbf{F} \cdot \mathbf{D} = 12 + 15 + 60 = 87$ J

14. $|\boldsymbol{\tau}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = (0.40)(50)\sin(90^\circ - 30^\circ) \approx 17.3$ N·m.

16. A direction vector for the line is $\mathbf{v} = \langle 3, 2, 1 \rangle$, so parametric equations for the line are $x = 1 + 3t$, $y = 2t$, $z = -1 + t$.

18.
Since the two planes are parallel, they will have the same normal vectors. Then we can take $\mathbf{n} = \langle 1, 4, -3 \rangle$ and an equation of the plane is $1(x - 2) + 4(y - 1) - 3(z - 0) = 0$ or $x + 4y - 3z = 6$.

20.
If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle 2, -1, 3 \rangle$ is one vector in the plane. We can verify that the given point $(1, 2, -2)$ does not lie on this line. The point $(0, 3, 1)$ is on the line (obtained by putting $t = 0$) and hence in the plane, so the vector $\mathbf{b} = \langle 0 - 1, 3 - 2, 1 - (-2) \rangle = \langle -1, 1, 3 \rangle$ lies in the plane, and a normal vector is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -6, -9, 1 \rangle$. Thus an equation of the plane is $-6(x - 1) - 9(y - 2) + (z + 2) = 0$ or $6x + 9y - z = 26$.

22. Use the formula proven in Exercise 12.4.45(a). In the notation used in that exercise, $\mathbf{a}$ is just the direction of the line; that is, $\mathbf{a} = \langle 1, -1, 2 \rangle$. A point on the line is $(1, 2, -1)$ (setting $t = 0$), and therefore $\mathbf{b} = \langle 1 - 0, 2 - 0, -1 - 0 \rangle = \langle 1, 2, -1 \rangle$.

Hence $d = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \dfrac{|\langle 1, -1, 2 \rangle \times \langle 1, 2, -1 \rangle|}{\sqrt{1 + 1 + 4}} = \dfrac{|\langle -3, 3, 3 \rangle|}{\sqrt{6}} = \sqrt{\dfrac{27}{6}} = \dfrac{3}{\sqrt{2}}$.

24. (a) The normal vectors are $\langle 1, 1, -1 \rangle$ and $\langle 2, -3, 4 \rangle$. Since these vectors aren't parallel, neither are the planes parallel. Also $\langle 1, 1, -1 \rangle \cdot \langle 2, -3, 4 \rangle = 2 - 3 - 4 = -5 \neq 0$ so the normal vectors, and thus the planes, are not perpendicular.

(b) $\cos\theta = \dfrac{\langle 1, 1, -1 \rangle \cdot \langle 2, -3, 4 \rangle}{\sqrt{3}\,\sqrt{29}} = -\dfrac{5}{\sqrt{87}}$ and $\theta = \cos^{-1}\left(-\frac{5}{\sqrt{87}}\right) \approx 122^\circ$ [or we can say $\approx 58^\circ$].

26.

(a) The vectors $\overrightarrow{AB} = \langle -1-2, -1-1, 10-1\rangle = \langle -3, -2, 9\rangle$ and $\overrightarrow{AC} = \langle 1-2, 3-1, -4-1\rangle = \langle -1, 2, -5\rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \langle -3, -2, 9\rangle \times \langle -1, 2, -5\rangle = \langle -8, -24, -8\rangle$ or equivalently $\langle 1, 3, 1\rangle$ is a normal vector to the plane. The point $A(2, 1, 1)$ lies on the plane so an equation of the plane is $1(x-2) + 3(y-1) + 1(z-1) = 0$ or $x + 3y + z = 6$.

(b) The line is perpendicular to the plane so it is parallel to a normal vector for the plane, namely $\langle 1, 3, 1\rangle$. If the line passes through $B(-1, -1, 10)$ then symmetric equations are $\dfrac{x-(-1)}{1} = \dfrac{y-(-1)}{3} = \dfrac{z-10}{1}$ or $x + 1 = \dfrac{y+1}{3} = z - 10$.

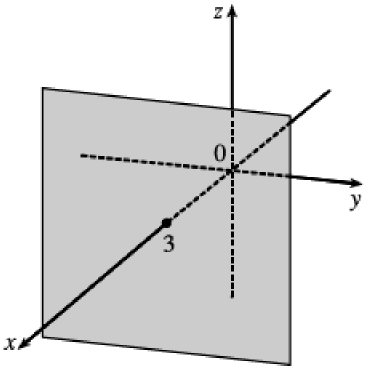
(c) Normal vectors for the two planes are $\mathbf{n}_1 = \langle 1, 3, 1\rangle$ and $\mathbf{n}_2 = \langle 2, -4, -3\rangle$. The angle θ between the planes is given by

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\ |\mathbf{n}_2|} = \frac{\langle 1, 3, 1\rangle \cdot \langle 2, -4, -3\rangle}{\sqrt{1^2+3^2+1^2}\,\sqrt{2^2+(-4)^2+(-3)^2}} = \frac{2-12-3}{\sqrt{11}\,\sqrt{29}} = -\frac{13}{\sqrt{319}}$$

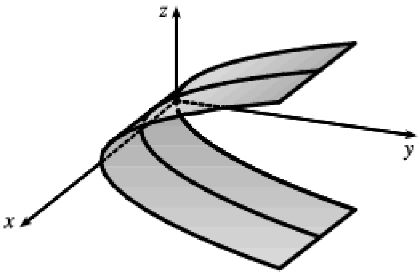
Thus $\theta = \cos^{-1}\left(-\dfrac{13}{\sqrt{319}}\right) \approx 137^\circ$ or $180^\circ - 137^\circ = 43^\circ$.

(d) From part (c), the point $(2, 0, 4)$ lies on the second plane, but notice that the point also satisfies the equation of the first plane, so the point lies on the line of intersection of the planes. A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes, so take $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 3, 1\rangle \times \langle 2, -4, -3\rangle = \langle -5, 5, -10\rangle$ or equivalently we can take $\mathbf{v} = \langle 1, -1, 2\rangle$. Parametric equations for the line are $x = 2 + t$, $y = -t$, $z = 4 + 2t$.

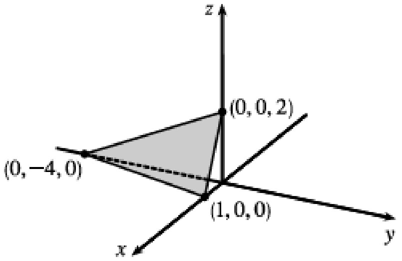
28. The equation $x = 3$ represents a plane parallel to the yz-plane and 3 units in front of it.



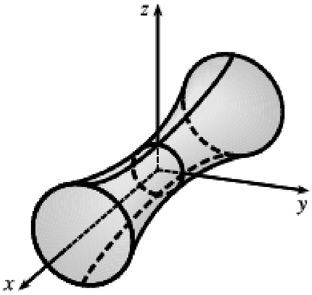
30. The equation $y = z^2$ represents a parabolic cylinder whose trace in the xz-plane is the x-axis and which opens to the right.



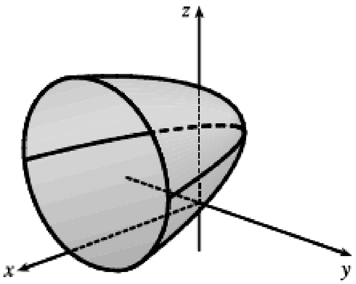
32. $4x - y + 2z = 4$ is a plane with intercepts $(1, 0, 0)$, $(0, -4, 0)$, and $(0, 0, 2)$.



34. An equivalent equation is $-x^2 + y^2 + z^2 = 1$, a hyperboloid of one sheet with axis the x-axis.



36. Completing the square in y and z gives $x = (y - 1)^2 + (z - 2)^2$, a circular paraboloid with vertex $(0, 1, 2)$ and axis the horizontal line $y = 1$, $z = 2$.



38. The distance from a point $P\,(x, y, z)$ to the plane $y = 1$ is $|y - 1|$, so the given condition becomes

$$|y-1| = 2\sqrt{(x-0)^2 + (y+1)^2 + (z-0)^2} \quad\Rightarrow\quad |y-1| = 2\sqrt{x^2 + (y+1)^2 + z^2} \quad\Rightarrow$$

$$(y-1)^2 = 4x^2 + 4(y+1)^2 + 4z^2 \quad\Leftrightarrow\quad -3 = 4x^2 + (3y^2 + 10y) + 4z^2 \quad\Leftrightarrow$$

$$\tfrac{16}{3} = 4x^2 + 3\left(y + \tfrac{5}{3}\right)^2 + 4z^2 \quad\Rightarrow\quad \tfrac{3}{4}x^2 + \tfrac{9}{16}\left(y + \tfrac{5}{3}\right)^2 + \tfrac{3}{4}z^2 = 1.$$

This is the equation of an ellipsoid whose center is $\left(0, -\frac{5}{3}, 0\right)$.