

1. (a) A function f of two variables is a rule that assigns to each ordered pair (x, y) of real numbers in its domain a unique real number denoted by $f(x, y)$.
(b) One way to visualize a function of two variables is by graphing it, resulting in the surface $z = f(x, y)$. Another method for visualizing a function of two variables is a contour map. The contour map consists of level curves of the function which are horizontal traces of the graph of the function projected onto the xy -plane. Also, we can use an arrow diagram such as Figure 1 in Section 14.1.
2. A function f of three variables is a rule that assigns to each ordered triple (x, y, z) in its domain a unique real number $f(x, y, z)$. We can visualize a function of three variables by examining its level surfaces $f(x, y, z) = k$, where k is a constant.
3. $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$ means the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that is within the domain of f . We can show that a limit at a point does not exist by finding two different paths approaching the point along which $f(x, y)$ has different limits.
4. (a) See Definition 14.2.4.
(b) If f is continuous on \mathbb{R}^2 , its graph will appear as a surface without holes or breaks.
5. (a) See (2) and (3) in Section 14.3.
(b) See “Interpretations of Partial Derivatives” on page 927 [ET 903].
(c) To find f_x , regard y as a constant and differentiate $f(x, y)$ with respect to x . To find f_y , regard x as a constant and differentiate $f(x, y)$ with respect to y .
6. See the statement of Clairaut’s Theorem on page 931 [ET 907].
7. (a) See (2) in Section 14.4.
(b) See (19) and the preceding discussion in Section 14.6.
8. See (3) and (4) and the accompanying discussion in Section 14.4. We can interpret the linearization of f at (a, b) geometrically as the linear function whose graph is the tangent plane to the graph of f at (a, b) . Thus it is the linear function which best approximates f near (a, b) .
9. (a) See Definition 14.4.7.
(b) Use Theorem 14.4.8.
10. See (10) and the associated discussion in Section 14.4.

11. See (2) and (3) in Section 14.5.

12. See (7) and the preceding discussion in Section 14.5.

1. True. $f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$ from Equation 14.3.3. Let $h = y - b$. As $h \rightarrow 0$, $y \rightarrow b$. Then by substituting,

$$\text{we get } f_y(a, b) = \lim_{y \rightarrow b} \frac{f(a, y) - f(a, b)}{y - b}.$$

2. False. If there were such a function, then $f_{xy} = 2y$ and $f_{yx} = 1$. So $f_{xy} \neq f_{yx}$, which contradicts Clairaut's Theorem.

3. False. $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$.

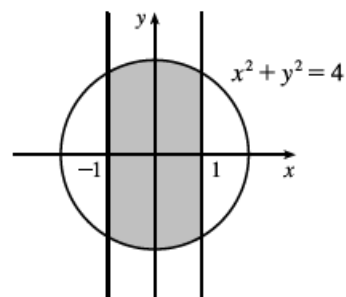
4. True. From Equation 14.6.14 we get $D_{\mathbf{k}} f(x, y, z) = \nabla f(x, y, z) \cdot \langle 0, 0, 1 \rangle = f_z(x, y, z)$.

5. False. See Example 14.2.3.

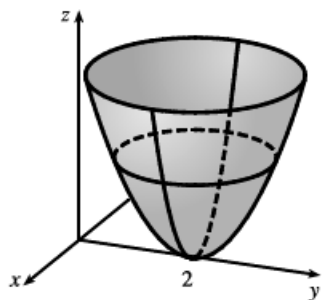
6. False. See Exercise 14.4.46(a).

8. False. If f is not continuous at $(2, 5)$, then we can have $\lim_{(x,y) \rightarrow (2,5)} f(x, y) \neq f(2, 5)$. (See Example 14.2.7)

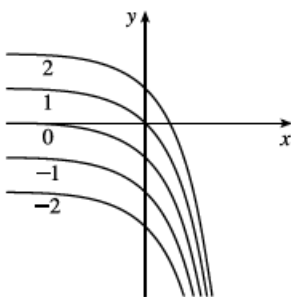
2. $\sqrt{4 - x^2 - y^2}$ is defined only when $4 - x^2 - y^2 \geq 0 \Leftrightarrow x^2 + y^2 \leq 4$, and $\sqrt{1 - x^2}$ is defined only when $1 - x^2 \geq 0 \Leftrightarrow -1 \leq x \leq 1$, so the domain of f is $\{(x, y) \mid -1 \leq x \leq 1, -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}\}$, which consists of those points on or inside the circle $x^2 + y^2 = 4$ for $-1 \leq x \leq 1$.



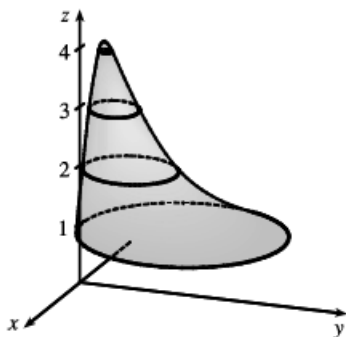
4. $z = f(x, y) = x^2 + (y - 2)^2$, a circular paraboloid with vertex $(0, 2, 0)$ and axis parallel to the z -axis



6. The level curves are $e^x + y = k$ or $y = -e^x + k$, a family of exponential curves.



8.



10. As $(x, y) \rightarrow (0, 0)$ along the x -axis, $f(x, 0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x, y) \rightarrow 0$ along this line. But $f(x, x) = 2x^2/(3x^2) = \frac{2}{3}$, so as $(x, y) \rightarrow (0, 0)$ along the line $x = y$, $f(x, y) \rightarrow \frac{2}{3}$. Thus the limit doesn't exist.
12. From the table, $T(6, 4) = 80$, and from Exercise 11 we estimated $T_x(6, 4) \approx 3.5$ and $T_y(6, 4) \approx -3.0$. The linear approximation then is

$$T(x, y) \approx T(6, 4) + T_x(6, 4)(x - 6) + T_y(6, 4)(y - 4) \approx 80 + 3.5(x - 6) - 3(y - 4) = 3.5x - 3y + 71$$

Thus at the point $(5, 3.8)$, we can use the linear approximation to estimate $T(5, 3.8) \approx 3.5(5) - 3(3.8) + 71 \approx 77.1^\circ\text{C}$.

$$14. g(u, v) = \frac{u + 2v}{u^2 + v^2} \Rightarrow g_u = \frac{(u^2 + v^2)(1) - (u + 2v)(2u)}{(u^2 + v^2)^2} = \frac{v^2 - u^2 - 4uv}{(u^2 + v^2)^2},$$

$$g_v = \frac{(u^2 + v^2)(2) - (u + 2v)(2v)}{(u^2 + v^2)^2} = \frac{2u^2 - 2v^2 - 2uv}{(u^2 + v^2)^2}$$

$$16. G(x, y, z) = e^{xz} \sin(y/z) \Rightarrow G_x = ze^{xz} \sin(y/z), G_y = e^{xz} \cos(y/z)(1/z) = (e^{xz}/z) \cos(y/z),$$

$$G_z = e^{xz} \cdot \cos(y/z)(-y/z^2) + \sin(y/z) \cdot xe^{xz} = e^{xz} [x \sin(y/z) - (y/z^2) \cos(y/z)]$$

$$18. C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) + 0.016D \Rightarrow$$

$\partial C/\partial T = 4.6 - 0.11T + 0.00087T^2 - 0.01(S - 35)$, $\partial C/\partial S = 1.34 - 0.01T$, and $\partial C/\partial D = 0.016$. When $T = 10$, $S = 35$, and $D = 100$ we have $\partial C/\partial T = 4.6 - 0.11(10) + 0.00087(10)^2 - 0.01(35 - 35) \approx 3.587$, thus in 10°C water with salinity 35 parts per thousand and a depth of 100 m, the speed of sound increases by about 3.59 m/s for every degree Celsius that the water temperature rises. Similarly, $\partial C/\partial S = 1.34 - 0.01(10) = 1.24$, so the speed of sound increases by about 1.24 m/s for every part per thousand the salinity of the water increases. $\partial C/\partial D = 0.016$, so the speed of sound increases by about 0.016 m/s for every meter that the depth is increased.

$$20. z = xe^{-2y} \Rightarrow z_x = e^{-2y}, z_y = -2xe^{-2y}, z_{xx} = 0, z_{yy} = 4xe^{-2y}, z_{xy} = z_{yx} = -2e^{-2y}$$

22.

$$v = r \cos(s + 2t) \Rightarrow v_r = \cos(s + 2t), v_s = -r \sin(s + 2t), v_t = -2r \sin(s + 2t), v_{rr} = 0, v_{ss} = -r \cos(s + 2t),$$

$$v_{tt} = -4r \cos(s + 2t), v_{rs} = v_{sr} = -\sin(s + 2t), v_{rt} = v_{tr} = -2 \sin(s + 2t), v_{st} = v_{ts} = -2r \cos(s + 2t)$$

$$24. z = \sin(x + \sin t) \Rightarrow \frac{\partial z}{\partial x} = \cos(x + \sin t), \frac{\partial z}{\partial t} = \cos(x + \sin t) \cos t,$$

$$\frac{\partial^2 z}{\partial x \partial t} = -\sin(x + \sin t) \cos t, \frac{\partial^2 z}{\partial x^2} = -\sin(x + \sin t) \text{ and}$$

$$\frac{\partial z}{\partial x} \frac{\partial^2 z}{\partial x \partial t} = \cos(x + \sin t) [-\sin(x + \sin t) \cos t] = \cos(x + \sin t) (\cos t) [-\sin(x + \sin t)] = \frac{\partial z}{\partial t} \frac{\partial^2 z}{\partial x^2}.$$

$$26. (a) z_x = e^x \cos y \Rightarrow z_x(0, 0) = 1 \text{ and } z_y = -e^x \sin y \Rightarrow z_y(0, 0) = 0, \text{ so an equation of the tangent plane is}$$

$$z - 1 = 1(x - 0) + 0(y - 0) \text{ or } z = x + 1.$$

(b) A normal vector to the tangent plane (and the surface) at $(0, 0, 1)$ is $\langle 1, 0, -1 \rangle$. Then parametric equations for the normal line there are $x = t, y = 0, z = 1 - t$, and symmetric equations are $x = 1 - z, y = 0$.

$$28. (a) \text{ Let } F(x, y, z) = xy + yz + zx. \text{ Then } F_x = y + z, F_y = x + z, F_z = x + y, \text{ so}$$

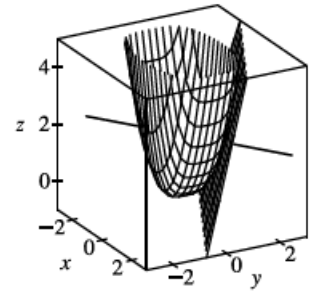
$$F_x(1, 1, 1) = F_y(1, 1, 1) = F_z(1, 1, 1) = 2. \text{ From Equation 14.6.19, an equation of the tangent plane is}$$

$$2(x - 1) + 2(y - 1) + 2(z - 1) = 0 \text{ or, equivalently, } x + y + z = 3.$$

$$(b) \text{ From Equations 14.6.20, symmetric equations for the normal line are } \frac{x - 1}{2} = \frac{y - 1}{2} = \frac{z - 1}{2} \text{ or, equivalently,}$$

$$x = y = z.$$

30. Let $f(x, y) = x^2 + y^4$. Then $f_x(x, y) = 2x$ and $f_y(x, y) = 4y^3$, so $f_x(1, 1) = 2$, $f_y(1, 1) = 4$ and an equation of the tangent plane is $z - 2 = 2(x - 1) + 4(y - 1)$ or $2x + 4y - z = 4$. A normal vector to the tangent plane is $\langle 2, 4, -1 \rangle$ so the normal line is given by $\frac{x-1}{2} = \frac{y-1}{4} = \frac{z-2}{-1}$ or $x = 1 + 2t$, $y = 1 + 4t$, $z = 2 - t$.



32. $u = \ln(1 + se^{2t}) \Rightarrow du = \frac{\partial u}{\partial s} ds + \frac{\partial u}{\partial t} dt = \frac{e^{2t}}{1 + se^{2t}} ds + \frac{2se^{2t}}{1 + se^{2t}} dt$

34.

(a) $dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy = \frac{1}{2}y dx + \frac{1}{2}x dy$ and $|\Delta x| \leq 0.002$, $|\Delta y| \leq 0.002$. Thus the maximum error in the calculated area is about $dA = 6(0.002) + \frac{5}{2}(0.002) = 0.017 \text{ m}^2$ or 170 cm^2 .

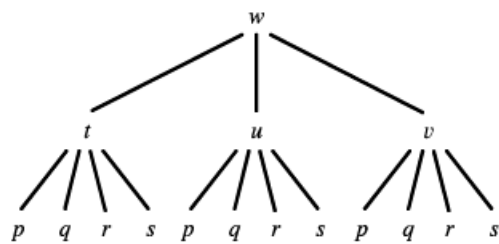
(b) $z = \sqrt{x^2 + y^2}$, $dz = \frac{x}{\sqrt{x^2 + y^2}} dx + \frac{y}{\sqrt{x^2 + y^2}} dy$ and $|\Delta x| \leq 0.002$, $|\Delta y| \leq 0.002$. Thus the maximum error in the calculated hypotenuse length is about $dz = \frac{5}{13}(0.002) + \frac{12}{13}(0.002) = \frac{0.17}{65} \approx 0.0026 \text{ m}$ or 0.26 cm .

36. $\frac{\partial v}{\partial s} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial s} = (2x \sin y + y^2 e^{xy})(1) + (x^2 \cos y + xye^{xy} + e^{xy})(t)$.

$s = 0, t = 1 \Rightarrow x = 2, y = 0$, so $\frac{\partial v}{\partial s} = 0 + (4 + 1)(1) = 5$.

$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial t} = (2x \sin y + y^2 e^{xy})(2) + (x^2 \cos y + xye^{xy} + e^{xy})(s) = 0 + 0 = 0$.

38.



Using the tree diagram as a guide, we have

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial t} \frac{\partial t}{\partial p} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial p} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial p} \quad \frac{\partial w}{\partial q} = \frac{\partial w}{\partial t} \frac{\partial t}{\partial q} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial q} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial q}$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial t} \frac{\partial t}{\partial r} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial r} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial r} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial t} \frac{\partial t}{\partial s} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial s}$$

40. $A = \frac{1}{2}xy \sin \theta$, $dx/dt = 3$, $dy/dt = -2$, $d\theta/dt = 0.05$, and $\frac{dA}{dt} = \frac{1}{2} \left[(y \sin \theta) \frac{dx}{dt} + (x \sin \theta) \frac{dy}{dt} + (xy \cos \theta) \frac{d\theta}{dt} \right]$.

So when $x = 40$, $y = 50$ and $\theta = \frac{\pi}{6}$, $\frac{dA}{dt} = \frac{1}{2} [(25)(3) + (20)(-2) + (1000 \sqrt{3})(0.05)] = \frac{35 + 50 \sqrt{3}}{2} \approx 60.8 \text{ in}^2/\text{s}$.

42. $\cos(xyz) = 1 + x^2y^2 + z^2$, so let $F(x, y, z) = 1 + x^2y^2 + z^2 - \cos(xyz) = 0$. Then by

$$\text{Equations 14.5.7 we have } \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2xy^2 + \sin(xyz) \cdot yz}{2z + \sin(xyz) \cdot xy} = -\frac{2xy^2 + yz \sin(xyz)}{2z + xy \sin(xyz)},$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2x^2y + \sin(xyz) \cdot xz}{2z + \sin(xyz) \cdot xy} = -\frac{2x^2y + xz \sin(xyz)}{2z + xy \sin(xyz)}.$$

44.

(a) By Theorem 14.6.15, the maximum value of the directional derivative occurs when \mathbf{u} has the same direction as the gradient vector.

(b) It is a minimum when \mathbf{u} is in the direction opposite to that of the gradient vector (that is, \mathbf{u} is in the direction of $-\nabla f$), since $D_{\mathbf{u}} f = |\nabla f| \cos \theta$ (see the proof of Theorem 14.6.15) has a minimum when $\theta = \pi$.

(c) The directional derivative is 0 when \mathbf{u} is perpendicular to the gradient vector, since then $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = 0$.

(d) The directional derivative is half of its maximum value when $D_{\mathbf{u}} f = |\nabla f| \cos \theta = \frac{1}{2} |\nabla f| \Leftrightarrow \cos \theta = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{3}$.

46. $\nabla f = \langle 2xy + \sqrt{1+z}, x^2, x/(2\sqrt{1+z}) \rangle$, $\nabla f(1, 2, 3) = \langle 6, 1, \frac{1}{4} \rangle$, $\mathbf{u} = \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \rangle$. Then $D_{\mathbf{u}} f(1, 2, 3) = \frac{25}{6}$.

48. $\nabla f = \langle zye^{xy}, zxe^{xy}, e^{xy} \rangle$, $\nabla f(0, 1, 2) = \langle 2, 0, 1 \rangle$ is the direction of most rapid increase while the rate is $|\langle 2, 0, 1 \rangle| = \sqrt{5}$.

50. The surfaces are $f(x, y, z) = z - 2x^2 + y^2 = 0$ and $g(x, y, z) = z - 4 = 0$. The tangent line is perpendicular to both ∇f and ∇g at $(-2, 2, 4)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ is therefore parallel to the line. $\nabla f(x, y, z) = \langle -4x, 2y, 1 \rangle \Rightarrow \nabla f(-2, 2, 4) = \langle 8, 4, 1 \rangle$, $\nabla g(x, y, z) = \langle 0, 0, 1 \rangle \Rightarrow \nabla g(-2, 2, 4) = \langle 0, 0, 1 \rangle$. Hence

$$\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4\mathbf{i} - 8\mathbf{j}. \text{ Thus, parametric equations are: } x = -2 + 4t, y = 2 - 8t, z = 4.$$

52.

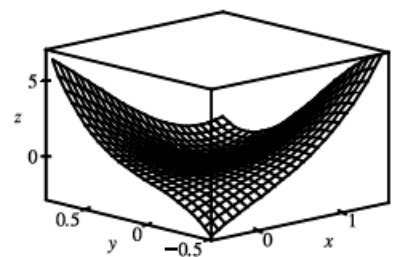
$$f(x, y) = x^3 - 6xy + 8y^3 \Rightarrow f_x = 3x^2 - 6y, f_y = -6x + 24y^2, f_{xx} = 6x,$$

$$f_{yy} = 48y, f_{xy} = -6. \text{ Then } f_x = 0 \text{ implies } y = x^2/2, \text{ substituting into } f_y = 0$$

implies $6x(x^3 - 1) = 0$, so the critical points are $(0, 0)$, $(1, \frac{1}{2})$.

$D(0, 0) = -36 < 0$ so $(0, 0)$ is a saddle point while $f_{xx}(1, \frac{1}{2}) = 6 > 0$ and

$D(1, \frac{1}{2}) = 108 > 0$ so $f(1, \frac{1}{2}) = -1$ is a local minimum.



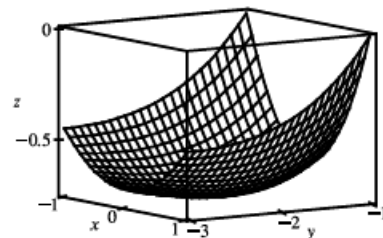
54.

$$f(x, y) = (x^2 + y)e^{y/2} \Rightarrow f_x = 2xe^{y/2}, f_y = e^{y/2}(2 + x^2 + y)/2,$$

$$f_{xx} = 2e^{y/2}, f_{yy} = e^{y/2}(4 + x^2 + y)/4, f_{xy} = xe^{y/2}. \text{ Then } f_x = 0 \text{ implies}$$

$$x = 0, \text{ so } f_y = 0 \text{ implies } y = -2. \text{ But } f_{xx}(0, -2) > 0, D(0, -2) = e^{-2} - 0 > 0$$

so $f(0, -2) = -2/e$ is a local minimum.



56. Inside D : $f_x = 2xe^{-x^2-y^2}(1 - x^2 - 2y^2) = 0$ implies $x = 0$ or $x^2 + 2y^2 = 1$. Then if $x = 0$,

$$f_y = 2ye^{-x^2-y^2}(2 - x^2 - 2y^2) = 0 \text{ implies } y = 0 \text{ or } 2 - 2y^2 = 0 \text{ giving the critical points } (0, 0), (0, \pm 1). \text{ If}$$

$$x^2 + 2y^2 = 1, \text{ then } f_y = 0 \text{ implies } y = 0 \text{ giving the critical points } (\pm 1, 0). \text{ Now } f(0, 0) = 0, f(\pm 1, 0) = e^{-1} \text{ and}$$

$$f(0, \pm 1) = 2e^{-1}. \text{ On the boundary of } D: x^2 + y^2 = 4, \text{ so } f(x, y) = e^{-4}(4 + y^2) \text{ and } f \text{ is smallest when } y = 0 \text{ and largest}$$

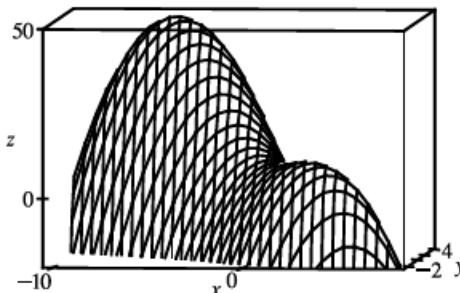
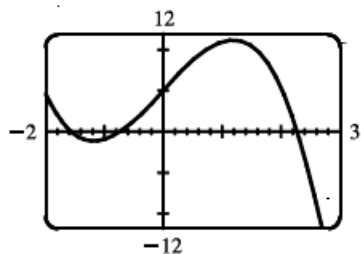
$$\text{when } y^2 = 4. \text{ But } f(\pm 2, 0) = 4e^{-4}, f(0, \pm 2) = 8e^{-4}. \text{ Thus on } D \text{ the absolute maximum of } f \text{ is } f(0, \pm 1) = 2e^{-1} \text{ and the}$$

absolute minimum is $f(0, 0) = 0$.

58.

$$f(x, y) = 12 + 10y - 2x^2 - 8xy - y^4 \Rightarrow f_x(x, y) = -4x - 8y, f_y(x, y) = 10 - 8x - 4y^3. \text{ Now } f_x(x, y) = 0 \Rightarrow$$

$$x = -2x, \text{ and substituting this into } f_y(x, y) = 0 \text{ gives } 10 + 16y - 4y^3 = 0 \Leftrightarrow 5 + 8y - 2y^3 = 0.$$



From the first graph, we see that this is true when $y \approx -1.542, -0.717, \text{ or } 2.260$. (Alternatively, we could have found the solutions to $f_x = f_y = 0$ using a CAS.) So to three decimal places, the critical points are $(3.085, -1.542)$, $(1.434, -0.717)$, and $(-4.519, 2.260)$. Now in order to use the Second Derivatives Test, we calculate $f_{xx} = -4$, $f_{xy} = -8$, $f_{yy} = -12y^2$, and $D = 48y^2 - 64$. So since $D(3.085, -1.542) > 0$, $D(1.434, -0.717) < 0$, and $D(-4.519, 2.260) > 0$, and f_{xx} is always negative, $f(x, y)$ has local maxima $f(-4.519, 2.260) \approx 49.373$ and $f(3.085, -1.542) \approx 9.948$, and a saddle point at approximately $(1.434, -0.717)$. The highest point on the graph is approximately $(-4.519, 2.260, 49.373)$.

60.

$f(x, y) = 1/x + 1/y$, $g(x, y) = 1/x^2 + 1/y^2 = 1 \Rightarrow \nabla f = \langle -x^{-2}, -y^{-2} \rangle = \lambda \nabla g = \langle -2\lambda x^{-3}, -2\lambda y^{-3} \rangle$. Then $-x^{-2} = -2\lambda x^{-3}$ or $x = 2\lambda$ and $-y^{-2} = -2\lambda y^{-3}$ or $y = 2\lambda$. Thus $x = y$, so $1/x^2 + 1/y^2 = 2/x^2 = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $(\pm\sqrt{2}, \pm\sqrt{2})$. The absolute maximum of f subject to $x^{-2} + y^{-2} = 1$ is then $f(\sqrt{2}, \sqrt{2}) = \sqrt{2}$ and the absolute minimum is $f(-\sqrt{2}, -\sqrt{2}) = -\sqrt{2}$.

62.

$f(x, y, z) = x^2 + 2y^2 + 3z^2$, $g(x, y, z) = x + y + z = 1$, $h(x, y, z) = x - y + 2z = 2 \Rightarrow \nabla f = \langle 2x, 4y, 6z \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda + \mu, \lambda - \mu, \lambda + 2\mu \rangle$ and $2x = \lambda + \mu$ (1), $4y = \lambda - \mu$ (2), $6z = \lambda + 2\mu$ (3), $x + y + z = 1$ (4), $x - y + 2z = 2$ (5). Then six times (1) plus three times (2) plus two times (3) implies $12(x + y + z) = 11\lambda + 7\mu$, so (4) gives $11\lambda + 7\mu = 12$. Also six times (1) minus three times (2) plus four times (3) implies $12(x - y + 2z) = 7\lambda + 17\mu$, so (5) gives $7\lambda + 17\mu = 24$. Solving $11\lambda + 7\mu = 12$, $7\lambda + 17\mu = 24$ simultaneously gives $\lambda = \frac{6}{23}$, $\mu = \frac{30}{23}$. Substituting into (1), (2), and (3) implies $x = \frac{18}{23}$, $y = -\frac{6}{23}$, $z = \frac{11}{23}$ giving only one point. Then $f(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}) = \frac{33}{23}$. Now since $(0, 0, 1)$ satisfies both constraints and $f(0, 0, 1) = 3 > \frac{33}{23}$, $f(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}) = \frac{33}{23}$ is an absolute minimum, and there is no absolute maximum.

64.

$V = xyz$, say x is the length and $x + 2y + 2z \leq 108$, $x > 0$, $y > 0$, $z > 0$. First maximize V subject to $x + 2y + 2z = 108$ with x, y, z all positive. Then $\langle yz, xz, xy \rangle = \langle \lambda, 2\lambda, 2\lambda \rangle$ implies $2yz = xz$ or $x = 2y$ and $xz = xy$ or $z = y$. Thus $g(x, y, z) = 108$ implies $6y = 108$ or $y = 18 = z$, $x = 36$, so the volume is $V = 11,664$ cubic units. Since $(104, 1, 1)$ also satisfies $g(x, y, z) = 108$ and $V(104, 1, 1) = 104$ cubic units, $(36, 18, 18)$ gives an absolute maximum of V subject to $g(x, y, z) = 108$. But if $x + 2y + 2z < 108$, there exists $\alpha > 0$ such that $x + 2y + 2z = 108 - \alpha$ and as above $6y = 108 - \alpha$ implies $y = (108 - \alpha)/6 = z$, $x = (108 - \alpha)/3$ with $V = (108 - \alpha)^3/(6^2 \cdot 3) < (108)^3/(6^2 \cdot 3) = 11,664$. Hence we have shown that the maximum of V subject to $g(x, y, z) \leq 108$ is the maximum of V subject to $g(x, y, z) = 108$ (an intuitively obvious fact).

66. (a) $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + f(x(t), y(t))\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt}\right)\mathbf{k}$

(by the Chain Rule). Therefore

$$K = \frac{1}{2}m|\mathbf{v}|^2 = \frac{m}{2} \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt}\right)^2 \right]$$

$$= \frac{m}{2} \left[(1 + f_x^2) \left(\frac{dx}{dt}\right)^2 + 2f_x f_y \left(\frac{dx}{dt}\right) \left(\frac{dy}{dt}\right) + (1 + f_y^2) \left(\frac{dy}{dt}\right)^2 \right]$$

(b) $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \left[f_{xx} \left(\frac{dx}{dt}\right)^2 + 2f_{xy} \frac{dx}{dt} \frac{dy}{dt} + f_{yy} \left(\frac{dy}{dt}\right)^2 + f_x \frac{d^2x}{dt^2} + f_y \frac{d^2y}{dt^2} \right]\mathbf{k}$

(c) If $z = x^2 + y^2$, where $x = t \cos t$ and $y = t \sin t$, then $z = f(x, y) = t^2$.

$$\mathbf{r} = t \cos t \mathbf{i} + t \sin t \mathbf{j} + t^2 \mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t \sin t) \mathbf{i} + (\sin t + t \cos t) \mathbf{j} + 2t \mathbf{k},$$

$$K = \frac{m}{2} [(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + (2t)^2] = \frac{m}{2} (1 + t^2 + 4t^2) = \frac{m}{2} (1 + 5t^2), \text{ and}$$

$$\mathbf{a} = (-2 \sin t - t \cos t) \mathbf{i} + (2 \cos t - t \sin t) \mathbf{j} + 2 \mathbf{k}. \text{ Notice that it is easier not to use the formulas in (a) and (b).}$$