1. (a) A double Riemann sum of f is $\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$, where ΔA is the area of each subrectangle and $\left(x_{ij}^*, y_{ij}^*\right)$ is a sample point in each subrectangle. If $f(x, y) \geq 0$, this sum represents an approximation to the volume of the solid that lies above the rectangle R and below the graph of f.

(b) $\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$

(c) If $f(x, y) \geq 0$, $\iint_R f(x, y)\, dA$ represents the volume of the solid that lies above the rectangle R and below the surface $z = f(x, y)$. If f takes on both positive and negative values, $\iint_R f(x, y)\, dA$ is the difference of the volume above R but below the surface $z = f(x, y)$ and the volume below R but above the surface $z = f(x, y)$.

(d) We usually evaluate $\iint_R f(x, y)\, dA$ as an iterated integral according to Fubini's Theorem (see Theorem 15.2.4).

(e) The Midpoint Rule for Double Integrals says that we approximate the double integral $\iint_R f(x, y)\, dA$ by the double Riemann sum $\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\overline{x}_i, \overline{y}_j\right) \Delta A$ where the sample points $\left(\overline{x}_i, \overline{y}_j\right)$ are the centers of the subrectangles.

(f) $f_{\text{ave}} = \dfrac{1}{A(R)} \displaystyle\iint_R f(x, y)\, dA$ where $A(R)$ is the area of R.

2. (a) See (1) and (2) and the accompanying discussion in Section 15.3.

(b) See (3) and the accompanying discussion in Section 15.3.

(c) See (5) and the preceding discussion in Section 15.3.

(d) See (6)–(11) in Section 15.3.

3. We may want to change from rectangular to polar coordinates in a double integral if the region R of integration is more easily described in polar coordinates. To accomplish this, we use $\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$ where R is given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$.

4. (a) $m = \iint_D \rho(x, y)\, dA$

(b) $M_x = \iint_D y\rho(x, y)\, dA$, $M_y = \iint_D x\rho(x, y)\, dA$

(c) The center of mass is $(\overline{x}, \overline{y})$ where $\overline{x} = \dfrac{M_y}{m}$ and $\overline{y} = \dfrac{M_x}{m}$.

(d) $I_x = \iint_D y^2\rho(x, y)\, dA,\ I_y = \iint_D x^2\rho(x, y)\, dA,\ I_0 = \iint_D (x^2 + y^2)\rho(x, y)\, dA$

5. (a) $P(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$

(b) $f(x,y) \ge 0$ and $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$.

(c) The expected value of X is $\mu_1 = \iint_{\mathbb{R}^2} x f(x,y)\,dA$; the expected value of Y is $\mu_2 = \iint_{\mathbb{R}^2} y f(x,y)\,dA$.

6. $A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\,dA$

7. (a) $\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right) \Delta V$

(b) We usually evaluate $\iiint_B f(x,y,z)\,dV$ as an iterated integral according to Fubini's Theorem for Triple Integrals (see Theorem 15.7.4).

(c) See the paragraph following Example 15.7.1.

(d) See (5) and (6) and the accompanying discussion in Section 15.7.

(e) See (10) and the accompanying discussion in Section 15.7.

(f) See (11) and the preceding discussion in Section 15.7.

8. (a) $m = \iiint_E \rho(x,y,z)\,dV$

(b) $M_{yz} = \iiint_E x\rho(x,y,z)\,dV,\ M_{xz} = \iiint_E y\rho(x,y,z)\,dV,\ M_{xy} = \iiint_E z\rho(x,y,z)\,dV$.

(c) The center of mass is $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = \dfrac{M_{yz}}{m}$, $\overline{y} = \dfrac{M_{xz}}{m}$, and $\overline{z} = \dfrac{M_{xy}}{m}$.

(d) $I_x = \iiint_E (y^2+z^2)\rho(x,y,z)\,dV,\ I_y = \iiint_E (x^2+z^2)\rho(x,y,z)\,dV,\ I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV$.

9. (a) See Formula 15.8.4 and the accompanying discussion.

(b) See Formula 15.9.3 and the accompanying discussion.

(c) We may want to change from rectangular to cylindrical or spherical coordinates in a triple integral if the region E of integration is more easily described in cylindrical or spherical coordinates or if the triple integral is easier to evaluate using cylindrical or spherical coordinates.

10. (a) $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}$

(b) See (9) and the accompanying discussion in Section 15.10.

(c) See (13) and the accompanying discussion in Section 15.10.

1. This is true by Fubini's Theorem.

2. False. $\int_0^1 \int_0^x \sqrt{x+y^2}\,dy\,dx$ describes the region of integration as a Type I region. To reverse the order of integration, we must consider the region as a Type II region: $\int_0^1 \int_y^1 \sqrt{x+y^2}\,dx\,dy$.

3. True by Equation 15.2.5.

4. $\int_{-1}^1 \int_0^1 e^{x^2+y^2} \sin y\,dx\,dy = \left(\int_0^1 e^{x^2}\,dx\right)\left(\int_{-1}^1 e^{y^2} \sin y\,dy\right) = \left(\int_0^1 e^{x^2}\,dx\right)(0) = 0$, since $e^{y^2}\sin y$ is an odd function. Therefore the statement is true.

5. True. By Equation 15.2.5 we can write $\int_0^1 \int_0^1 f(x)\,f(y)\,dy\,dx = \int_0^1 f(x)\,dx \int_0^1 f(y)\,dy$. But $\int_0^1 f(y)\,dy = \int_0^1 f(x)\,dx$ so this becomes $\int_0^1 f(x)\,dx \int_0^1 f(x)\,dx = \left[\int_0^1 f(x)\,dx\right]^2$.

6. This statement is true because in the given region, $\left(x^2+\sqrt{y}\right)\sin(x^2y^2) \le (1+2)(1) = 3$, so $\int_1^4 \int_0^1 \left(x^2+\sqrt{y}\right)\sin(x^2y^2)\,dx\,dy \le \int_1^4 \int_0^1 3\,dA = 3A(D) = 3(3) = 9$.

7. True: $\iint_D \sqrt{4-x^2-y^2}\,dA =$ the volume under the surface $x^2+y^2+z^2=4$ and above the xy-plane $= \frac{1}{2}\left(\text{the volume of the sphere } x^2+y^2+z^2=4\right) = \frac{1}{2}\cdot\frac{4}{3}\pi(2)^3 = \frac{16}{3}\pi$

8. True. The moment of inertia about the z-axis of a solid E with constant density k is $I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV = \iiint_E (kr^2)\,r\,dz\,dr\,d\theta = \iiint_E kr^3\,dz\,dr\,d\theta$.

9. The volume enclosed by the cone $z=\sqrt{x^2+y^2}$ and the plane $z=2$ is, in cylindrical coordinates, $V = \int_0^{2\pi}\int_0^2\int_r^2 r\,dz\,dr\,d\theta \ne \int_0^{2\pi}\int_0^2\int_r^2 dz\,dr\,d\theta$, so the assertion is false.

1. As shown in the contour map, we divide R into 9 equally sized subsquares, each with area $\Delta A = 1$. Then we approximate $\iint_R f(x,y)\,dA$ by a Riemann sum with $m=n=3$ and the sample points the upper right corners of each square, so

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{3}\sum_{j=1}^{3} f(x_i,y_j)\,\Delta A$$

$$= \Delta A\,[f(1,1)+f(1,2)+f(1,3)+f(2,1)+f(2,2)+f(2,3)+f(3,1)+f(3,2)+f(3,3)]$$

Using the contour lines to estimate the function values, we have

$$\iint_R f(x,y)\,dA \approx 1[2.7+4.7+8.0+4.7+6.7+10.0+6.7+8.6+11.9] \approx 64.0$$

2. As in Exercise 1, we have $m = n = 3$ and $\Delta A = 1$. Using the contour map to estimate the value of f at the center of each subsquare, we have

$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{3}\sum_{j=1}^{3} f(\overline{x}_i,\overline{y}_j)\,\Delta A\\ &= \Delta A\,[f(0.5,0.5) + (0.5,1.5) + (0.5,2.5) + (1.5,0.5) + f(1.5,1.5)\\ &\qquad + f(1.5,2.5) + (2.5,0.5) + f(2.5,1.5) + f(2.5,2.5)]\\ &\approx 1[1.2 + 2.5 + 5.0 + 3.2 + 4.5 + 7.1 + 5.2 + 6.5 + 9.0] = 44.2\end{aligned}$$

3. $\int_1^2\int_0^2 (y + 2xe^y)\,dx\,dy = \int_1^2 \left[xy + x^2e^y\right]_{x=0}^{x=2} dy = \int_1^2 (2y + 4e^y)\,dy = \left[y^2 + 4e^y\right]_1^2$
$= 4 + 4e^2 - 1 - 4e = 4e^2 - 4e + 3$

4. $\int_0^1\int_0^1 ye^{xy}\,dx\,dy = \int_0^1 \left[e^{xy}\right]_{x=0}^{x=1} dy = \int_0^1 (e^y - 1)\,dy = \left[e^y - y\right]_0^1 = e - 2$

5. $\int_0^1\int_0^x \cos(x^2)\,dy\,dx = \int_0^1 \left[\cos(x^2)y\right]_{y=0}^{y=x} dx = \int_0^1 x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2)\Big]_0^1 = \frac{1}{2}\sin 1$

6. $\int_0^1\int_x^{e^x} 3xy^2\,dy\,dx = \int_0^1 \left[xy^3\right]_{y=x}^{y=e^x} dx = \int_0^1 (xe^{3x} - x^4)\,dx = \frac{1}{3}xe^{3x}\Big]_0^1 - \int_0^1 \frac{1}{3}e^{3x}\,dx - \left[\frac{1}{5}x^5\right]_0^1$ [integrate by parts in the first term]
$= \frac{1}{3}e^3 - \left[\frac{1}{9}e^{3x}\right]_0^1 - \frac{1}{5} = \frac{2}{9}e^3 - \frac{4}{45}$

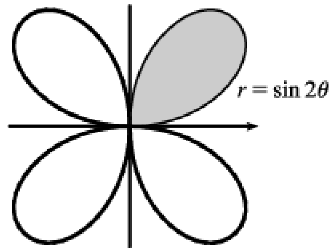
7. $\int_0^\pi\int_0^1\int_0^{\sqrt{1-y^2}} y\sin x\,dz\,dy\,dx = \int_0^\pi\int_0^1 \left[(y\sin x)z\right]_{z=0}^{z=\sqrt{1-y^2}} dy\,dx = \int_0^\pi\int_0^1 y\sqrt{1-y^2}\sin x\,dy\,dx$
$= \int_0^\pi \left[-\frac{1}{3}(1-y^2)^{3/2}\sin x\right]_{y=0}^{y=1} dx = \int_0^\pi \frac{1}{3}\sin x\,dx = -\frac{1}{3}\cos x\Big]_0^\pi = \frac{2}{3}$

8. $\int_0^1\int_0^y\int_x^1 6xyz\,dz\,dx\,dy = \int_0^1\int_0^y \left[3xyz^2\right]_{z=x}^{z=1} dx\,dy = \int_0^1\int_0^y (3xy - 3x^3y)\,dx\,dy$
$= \int_0^1 \left[\frac{3}{2}x^2y - \frac{3}{4}x^4y\right]_{x=0}^{x=y} dy = \int_0^1 \left(\frac{3}{2}y^3 - \frac{3}{4}y^5\right) dy = \left[\frac{3}{8}y^4 - \frac{1}{8}y^6\right]_0^1 = \frac{1}{4}$

9. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 2 \le r \le 4,\ 0 \le \theta \le \pi\}$. Thus

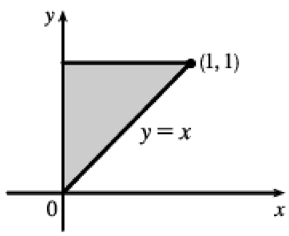
$\iint_R f(x,y)\,dA = \int_0^\pi\int_2^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

10. The region R is a type II region that can be described as the region enclosed by the lines $y = 4 - x$, $y = 4 + x$, and the x-axis. So using rectangular coordinates, we can say $R = \{(x,y) \mid y - 4 \le x \le 4 - y, 0 \le y \le 4\}$ and $\iint_R f(x,y)\,dA = \int_0^4\int_{y-4}^{4-y} f(x,y)\,dx\,dy$.

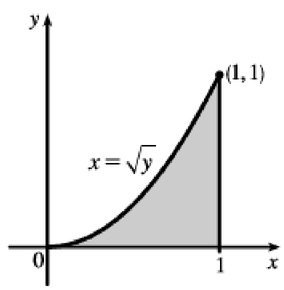
11. 

The region whose area is given by $\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta$ is $\left\{(r,\theta)\mid 0\le\theta\le\frac{\pi}{2}, 0\le r\le\sin 2\theta\right\}$, which is the region contained in the loop in the first quadrant of the four-leaved rose $r=\sin 2\theta$.

12. The solid is $\left\{(\rho,\theta,\phi)\mid 1\le\rho\le 2, 0\le\theta\le\frac{\pi}{2}, 0\le\phi\le\frac{\pi}{2}\right\}$ which is the region in the first octant on or between the two spheres $\rho=1$ and $\rho=2$.

13. 

$$\int_0^1\int_x^1 \cos(y^2)\,dy\,dx = \int_0^1\int_0^y \cos(y^2)\,dx\,dy = \int_0^1 \cos(y^2)\left[x\right]_{x=0}^{x=y}\,dy = \int_0^1 y\cos(y^2)\,dy = \left[\tfrac{1}{2}\sin(y^2)\right]_0^1 = \tfrac{1}{2}\sin 1$$

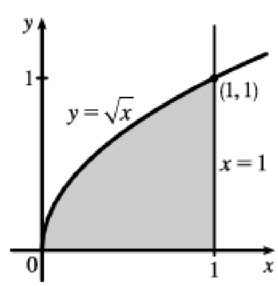
14. 

$$\int_0^1\int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\,dx\,dy = \int_0^1\int_0^{x^2}\frac{ye^{x^2}}{x^3}\,dy\,dx = \int_0^1 \frac{e^{x^2}}{x^3}\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=x^2}\,dx = \int_0^1 \tfrac{1}{2}xe^{x^2}\,dx = \tfrac{1}{4}e^{x^2}\Big]_0^1 = \tfrac{1}{4}(e-1)$$

15. $\iint_R ye^{xy}\,dA = \int_0^3\int_0^2 ye^{xy}\,dx\,dy = \int_0^3\left[e^{xy}\right]_{x=0}^{x=2}\,dy = \int_0^3(e^{2y}-1)\,dy = \left[\frac{1}{2}e^{2y}-y\right]_0^3 = \frac{1}{2}e^6 - 3 - \frac{1}{2} = \frac{1}{2}e^6 - \frac{7}{2}$

16. $\iint_D xy\,dA = \int_0^1\int_{y^2}^{y+2} xy\,dx\,dy = \int_0^1 y\left[\frac{1}{2}x^2\right]_{x=y^2}^{x=y+2}\,dy = \frac{1}{2}\int_0^1 y((y+2)^2 - y^4)\,dy$

$$= \tfrac{1}{2}\int_0^1 (y^3+4y^2+4y-y^5)\,dy = \tfrac{1}{2}\left[\tfrac{1}{4}y^4+\tfrac{4}{3}y^3+2y^2-\tfrac{1}{6}y^6\right]_0^1 = \tfrac{41}{24}$$

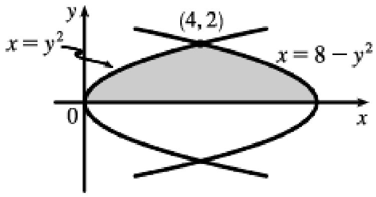
17. 

$$\iint_D \frac{y}{1+x^2}\,dA = \int_0^1\int_0^{\sqrt{x}}\frac{y}{1+x^2}\,dy\,dx = \int_0^1\frac{1}{1+x^2}\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=\sqrt{x}}\,dx = \tfrac{1}{2}\int_0^1\frac{x}{1+x^2}\,dx = \left[\tfrac{1}{4}\ln(1+x^2)\right]_0^1 = \tfrac{1}{4}\ln 2$$

18. $\iint_D \frac{1}{1+x^2}\,dA = \int_0^1 \int_x^1 \frac{1}{1+x^2}\,dy\,dx = \int_0^1 \frac{1}{1+x^2}\,[\,y\,]_{y=x}^{y=1}\,dx = \int_0^1 \frac{1-x}{1+x^2}\,dx = \int_0^1 \left(\frac{1}{1+x^2} - \frac{x}{1+x^2}\right)dx$

$$= \left[\tan^{-1} x - \tfrac{1}{2}\ln(1+x^2)\right]_0^1 = \tan^{-1} 1 - \tfrac{1}{2}\ln 2 - \left(\tan^{-1} 0 - \tfrac{1}{2}\ln 1\right) = \tfrac{\pi}{4} - \tfrac{1}{2}\ln 2$$

19.

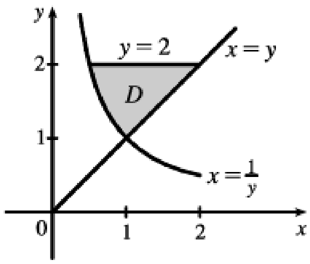


$$\iint_D y\,dA = \int_0^2 \int_{y^2}^{8-y^2} y\,dx\,dy$$

$$= \int_0^2 y\big[x\big]_{x=y^2}^{x=8-y^2}\,dy = \int_0^2 y(8-y^2-y^2)\,dy$$

$$= \int_0^2 (8y - 2y^3)\,dy = \left[4y^2 - \tfrac{1}{2}y^4\right]_0^2 = 8$$

20.

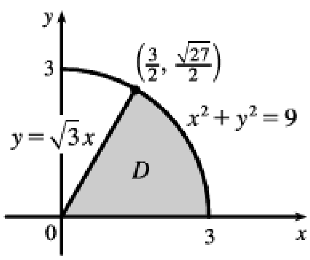


$$\iint_D y\,dA = \int_1^2 \int_{1/y}^{y} y\,dx\,dy = \int_1^2 y\left(y - \frac{1}{y}\right)dy$$

$$= \int_1^2 \left(y^2 - 1\right)dy = \left[\tfrac{1}{3}y^3 - y\right]_1^2$$

$$= \left(\tfrac{8}{3} - 2\right) - \left(\tfrac{1}{3} - 1\right) = \tfrac{4}{3}$$

21.



$$\iint_D \left(x^2+y^2\right)^{3/2} dA = \int_0^{\pi/3} \int_0^3 (r^2)^{3/2} r\,dr\,d\theta$$

$$= \int_0^{\pi/3} d\theta \int_0^3 r^4\,dr = \big[\,\theta\,\big]_0^{\pi/3}\,\left[\tfrac{1}{5}r^5\right]_0^3$$

$$= \frac{\pi}{3}\,\frac{3^5}{5} = \frac{81\pi}{5}$$

22. $\iint_D x\,dA = \int_0^{\pi/2} \int_1^{\sqrt{2}} (r\cos\theta)\,r\,dr\,d\theta = \int_0^{\pi/2} \cos\theta\,d\theta \int_1^{\sqrt{2}} r^2\,dr = \big[\sin\theta\big]_0^{\pi/2}\,\left[\tfrac{1}{3}r^3\right]_1^{\sqrt{2}}$

$$= 1 \cdot \tfrac{1}{3}(2^{3/2} - 1) = \tfrac{1}{3}(2^{3/2} - 1)$$

23. $\iiint_E xy\,dV = \int_0^3 \int_0^x \int_0^{x+y} xy\,dz\,dy\,dx = \int_0^3 \int_0^x xy\,[\,z\,]_{z=0}^{z=x+y}\,dy\,dx = \int_0^3 \int_0^x xy(x+y)\,dy\,dx$

$$= \int_0^3 \int_0^x (x^2y + xy^2)\,dy\,dx = \int_0^3 \left[\tfrac{1}{2}x^2y^2 + \tfrac{1}{3}xy^3\right]_{y=0}^{y=x} dx = \int_0^3 \left(\tfrac{1}{2}x^4 + \tfrac{1}{3}x^4\right) dx$$

$$= \tfrac{5}{6}\int_0^3 x^4\,dx = \left[\tfrac{1}{6}x^5\right]_0^3 = \tfrac{81}{2} = 40.5$$

24. $\iiint_T xy\,dV = \int_0^{1/3}\int_0^{1-3x}\int_0^{1-3x-y} xy\,dz\,dy\,dx = \int_0^{1/3}\int_0^{1-3x} xy(1-3x-y)\,dy\,dx$

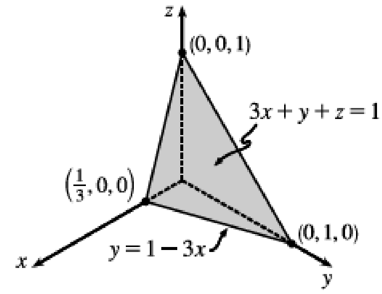
$= \int_0^{1/3}\int_0^{1-3x}(xy - 3x^2y - xy^2)\,dy\,dx$

$= \int_0^{1/3}\left[\frac{1}{2}xy^2 - \frac{3}{2}x^2y^2 - \frac{1}{3}xy^3\right]_{y=0}^{y=1-3x} dx$

$= \int_0^{1/3}\left[\frac{1}{2}x(1-3x)^2 - \frac{3}{2}x^2(1-3x)^2 - \frac{1}{3}x(1-3x)^3\right] dx$

$= \int_0^{1/3}\left(\frac{1}{6}x - \frac{3}{2}x^2 + \frac{9}{2}x^3 - \frac{9}{2}x^4\right) dx$

$= \left.\frac{1}{12}x^2 - \frac{1}{2}x^3 + \frac{9}{8}x^4 - \frac{9}{10}x^5\right]_0^{1/3} = \frac{1}{1080}$



25. $\iiint_E y^2z^2\,dV = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{1-y^2-z^2} y^2z^2\,dx\,dz\,dy = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^2z^2(1-y^2-z^2)\,dz\,dy$

$= \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta)(r^2\sin^2\theta)(1-r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{4}\sin^2 2\theta(r^5 - r^7)\,dr\,d\theta$

$= \int_0^{2\pi}\frac{1}{8}(1-\cos 4\theta)\left[\frac{1}{6}r^6 - \frac{1}{8}r^8\right]_{r=0}^{r=1} d\theta = \frac{1}{192}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{2\pi} = \frac{2\pi}{192} = \frac{\pi}{96}$

26. $\iiint_E z\,dV = \int_0^1\int_0^{\sqrt{1-y^2}}\int_0^{2-y} z\,dx\,dz\,dy = \int_0^1\int_0^{\sqrt{1-y^2}}(2-y)z\,dz\,dy = \int_0^1 \frac{1}{2}(2-y)(1-y^2)\,dy$

$= \int_0^1 \frac{1}{2}(2 - y - 2y^2 + y^3)\,dy = \frac{13}{24}$

27. $\iiint_E yz\,dV = \int_{-2}^{2}\int_0^{\sqrt{4-x^2}}\int_0^{y} yz\,dz\,dy\,dx = \int_{-2}^{2}\int_0^{\sqrt{4-x^2}} \frac{1}{2}y^3\,dy\,dx = \int_0^{\pi}\int_0^2 \frac{1}{2}r^3(\sin^3\theta)\,r\,dr\,d\theta$

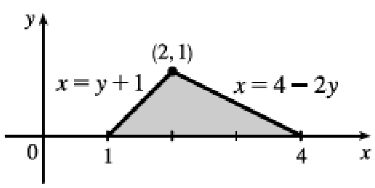
$= \frac{16}{5}\int_0^{\pi}\sin^3\theta\,d\theta = \frac{16}{5}\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi} = \frac{64}{15}$

28. $\iiint_H z^3\sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1(\rho^3\cos^3\phi)\rho(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$

$= \int_0^{2\pi}d\theta\,\int_0^{\pi/2}\cos^3\phi\sin\phi\,d\phi\,\int_0^1\rho^6\,d\rho = 2\pi\left[-\frac{1}{4}\cos^4\phi\right]_0^{\pi/2}\left(\frac{1}{7}\right) = \frac{\pi}{14}$

29. $V = \int_0^2\int_1^4(x^2+4y^2)\,dy\,dx = \int_0^2\left[x^2y + \frac{4}{3}y^3\right]_{y=1}^{y=4} dx = \int_0^2(3x^2+84)\,dx = 176$

30.

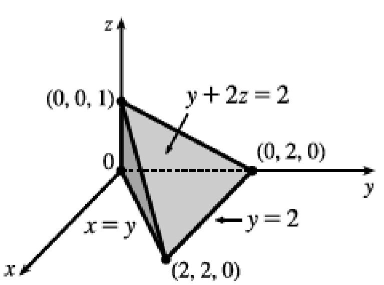


$V = \int_0^1\int_{y+1}^{4-2y}\int_0^{x^2y} dz\,dx\,dy = \int_0^1\int_{y+1}^{4-2y} x^2y\,dx\,dy$

$= \int_0^1 \frac{1}{3}\left[(4-2y)^3y - (y+1)^3y\right] dy$

$= \int_0^1 3(-y^4 + 5y^3 - 11y^2 + 7y)\,dy = 3\left(-\frac{1}{5} + \frac{5}{4} - \frac{11}{3} + \frac{7}{2}\right) = \frac{53}{20}$

31.



$V = \int_0^2\int_0^y\int_0^{(2-y)/2} dz\,dx\,dy = \int_0^2\int_0^y\left(1 - \frac{1}{2}y\right) dx\,dy$

$= \int_0^2\left(y - \frac{1}{2}y^2\right) dy = \frac{2}{3}$

32. $V = \int_0^{2\pi}\int_0^2\int_0^{3-r\sin\theta} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (3r - r^2\sin\theta)\,dr\,d\theta = \int_0^{2\pi}\left[6 - \frac{8}{3}\sin\theta\right]d\theta = 6\theta\big]_0^{2\pi} + 0 = 12\pi$

33. Using the wedge above the plane $z = 0$ and below the plane $z = mx$ and noting that we have the same volume for $m < 0$ as for $m > 0$ (so use $m > 0$), we have

$$V = 2\int_0^{a/3}\int_0^{\sqrt{a^2-9y^2}} mx\,dx\,dy = 2\int_0^{a/3}\tfrac{1}{2}m(a^2 - 9y^2)\,dy = m\left[a^2y - 3y^3\right]_0^{a/3} = m\left(\tfrac{1}{3}a^3 - \tfrac{1}{9}a^3\right) = \tfrac{2}{9}ma^3.$$

34. The paraboloid and the half-cone intersect when $x^2 + y^2 = \sqrt{x^2+y^2}$, that is when $x^2 + y^2 = 1$ or 0. So

$$V = \iint\limits_{x^2+y^2\le 1}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} dz\,dA = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r^2 - r^3)\,dr\,d\theta = \int_0^{2\pi}\left(\tfrac{1}{3} - \tfrac{1}{4}\right)d\theta = \tfrac{1}{12}(2\pi) = \tfrac{\pi}{6}.$$

35. (a) $m = \int_0^1\int_0^{1-y^2} y\,dx\,dy = \int_0^1 (y - y^3)\,dy = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

(b) $M_y = \int_0^1\int_0^{1-y^2} xy\,dx\,dy = \int_0^1 \frac{1}{2}y(1-y^2)^2\,dy = -\frac{1}{12}(1-y^2)^3\big]_0^1 = \frac{1}{12}$,

$M_x = \int_0^1\int_0^{1-y^2} y^2\,dx\,dy = \int_0^1 (y^2 - y^4)\,dy = \frac{2}{15}$. Hence $(\overline{x}, \overline{y}) = \left(\frac{1}{3}, \frac{8}{15}\right)$.

(c) $I_x = \int_0^1\int_0^{1-y^2} y^3\,dx\,dy = \int_0^1 (y^3 - y^5)\,dy = \frac{1}{12}$,

$I_y = \int_0^1\int_0^{1-y^2} yx^2\,dx\,dy = \int_0^1 \frac{1}{3}y(1-y^2)^3\,dy = -\frac{1}{24}(1-y^2)^4\big]_0^1 = \frac{1}{24}$,

$I_0 = I_x + I_y = \frac{1}{8}$, $\overline{\overline{y}}^2 = \frac{1/12}{1/4} = \frac{1}{3} \quad\Rightarrow\quad \overline{\overline{y}} = \frac{1}{\sqrt{3}}$, and $\overline{\overline{x}}^2 = \frac{1/24}{1/4} = \frac{1}{6} \quad\Rightarrow\quad \overline{\overline{x}} = \frac{1}{\sqrt{6}}$.

36. (a) $m = \frac{1}{4}\pi K a^2$ where K is constant,

$M_y = \iint_{x^2+y^2\le a^2} Kx\,dA = K\int_0^{\pi/2}\int_0^a r^2\cos\theta\,dr\,d\theta = \frac{1}{3}Ka^3\int_0^{\pi/2}\cos\theta\,d\theta = \frac{1}{3}a^3K$, and

$M_x = K\int_0^{\pi/2}\int_0^a r^2\sin\theta\,dr\,d\theta = \frac{1}{3}a^3K$ [by symmetry $M_y = M_x$].

Hence the centroid is $(\overline{x}, \overline{y}) = \left(\frac{4}{3\pi}a, \frac{4}{3\pi}a\right)$.

(b) $m = \int_0^{\pi/2}\int_0^a r^4\cos\theta\sin^2\theta\,dr\,d\theta = \left[\frac{1}{3}\sin^3\theta\right]_0^{\pi/2}\left(\frac{1}{5}a^5\right) = \frac{1}{15}a^5$,

$M_y = \int_0^{\pi/2}\int_0^a r^5\cos^2\theta\sin^2\theta\,dr\,d\theta = \frac{1}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2}\left(\frac{1}{6}a^6\right) = \frac{1}{96}\pi a^6$, and

$M_x = \int_0^{\pi/2}\int_0^a r^5\cos\theta\sin^3\theta\,dr\,d\theta = \left[\frac{1}{4}\sin^4\theta\right]_0^{\pi/2}\left(\frac{1}{6}a^6\right) = \frac{1}{24}a^6$. Hence $(\overline{x}, \overline{y}) = \left(\frac{5}{32}\pi a, \frac{5}{8}a\right)$.

37. (a) The equation of the cone with the suggested orientation is $(h - z) = \frac{h}{a}\sqrt{x^2 + y^2}$, $0 \le z \le h$. Then $V = \frac{1}{3}\pi a^2 h$ is the volume of one frustum of a cone; by symmetry $M_{yz} = M_{xz} = 0$; and

$$M_{xy} = \iint\limits_{x^2+y^2 \le a^2} \int_0^{h-(h/a)\sqrt{x^2+y^2}} z\,dz\,dA = \int_0^{2\pi}\int_0^a\int_0^{(h/a)(a-r)} rz\,dz\,dr\,d\theta = \pi\int_0^a r\frac{h^2}{a^2}(a-r)^2\,dr$$

$$= \frac{\pi h^2}{a^2}\int_0^a (a^2 r - 2ar^2 + r^3)\,dr = \frac{\pi h^2}{a^2}\left(\frac{a^4}{2} - \frac{2a^4}{3} + \frac{a^4}{4}\right) = \frac{\pi h^2 a^2}{12}$$

Hence the centroid is $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{1}{4}h\right)$.

(b) $$I_z = \int_0^{2\pi}\int_0^a\int_0^{(h/a)(a-r)} r^3\,dz\,dr\,d\theta = 2\pi\int_0^a \frac{h}{a}(ar^3 - r^4)\,dr = \frac{2\pi h}{a}\left(\frac{a^5}{4} - \frac{a^5}{5}\right) = \frac{\pi a^4 h}{10}$$

38. $1 \le z^2 \le 4 \quad\Rightarrow\quad 1/a^2 \le x^2 + y^2 \le 4/a^2$. Let $D = \{(x, y) \mid 1/a^2 \le x^2 + y^2 \le 4/a^2\}$. $z = f(x, y) = a\sqrt{x^2 + y^2}$, so $f_x(x, y) = ax(x^2 + y^2)^{-1/2}$, $f_y(x, y) = ay(x^2 + y^2)^{-1/2}$, and

$$A(S) = \iint_D \sqrt{\frac{a^2x^2 + a^2y^2}{x^2 + y^2} + 1}\,dA = \iint_D \sqrt{a^2 + 1}\,dA = \sqrt{a^2 + 1}\,A(D)$$

$$= \sqrt{a^2 + 1}\left[\pi\left(\frac{2}{a}\right)^2 - \pi\left(\frac{1}{a}\right)^2\right] = \frac{3\pi}{a^2}\sqrt{a^2 + 1}$$

39.
Let D represent the given triangle; then D can be described as the area enclosed by the x- and y-axes and the line $y = 2 - 2x$, or equivalently $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 2 - 2x\}$. We want to find the surface area of the part of the graph of $z = x^2 + y$ that lies over D, so using Equation 15.6.3 we have

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint_D \sqrt{1 + (2x)^2 + (1)^2}\,dA = \int_0^1\int_0^{2-2x} \sqrt{2 + 4x^2}\,dy\,dx$$

$$= \int_0^1 \sqrt{2 + 4x^2}\,[y]_{y=0}^{y=2-2x}\,dx = \int_0^1 (2 - 2x)\sqrt{2 + 4x^2}\,dx = \int_0^1 2\sqrt{2 + 4x^2}\,dx - \int_0^1 2x\sqrt{2 + 4x^2}\,dx$$

Using Formula 21 in the Table of Integrals with $a = \sqrt{2}$, $u = 2x$, and $du = 2\,dx$, we have

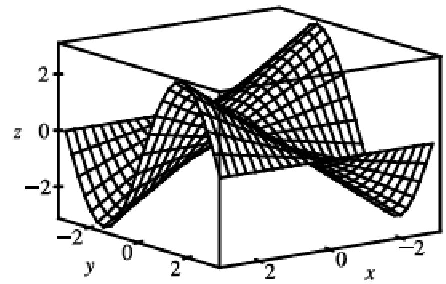
$\int 2\sqrt{2 + 4x^2}\,dx = x\sqrt{2 + 4x^2} + \ln\left(2x + \sqrt{2 + 4x^2}\right)$. If we substitute $u = 2 + 4x^2$ in the second integral, then $du = 8x\,dx$ and $\int 2x\sqrt{2 + 4x^2}\,dx = \frac{1}{4}\int \sqrt{u}\,du = \frac{1}{4}\cdot\frac{2}{3}u^{3/2} = \frac{1}{6}(2 + 4x^2)^{3/2}$. Thus

$$A(S) = \left[x\sqrt{2 + 4x^2} + \ln\left(2x + \sqrt{2 + 4x^2}\right) - \tfrac{1}{6}(2 + 4x^2)^{3/2}\right]_0^1$$

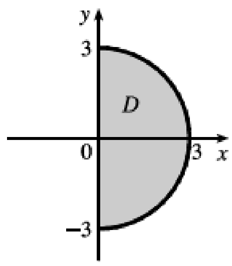
$$= \sqrt{6} + \ln\left(2 + \sqrt{6}\right) - \tfrac{1}{6}(6)^{3/2} - \ln\sqrt{2} + \tfrac{\sqrt{2}}{3} = \ln\tfrac{2 + \sqrt{6}}{\sqrt{2}} + \tfrac{\sqrt{2}}{3}$$

$$= \ln\left(\sqrt{2} + \sqrt{3}\right) + \tfrac{\sqrt{2}}{3} \approx 1.6176$$

40. Using Formula 15.6.3 with $\partial z/\partial x = \sin y$, $\partial z/\partial y = x\cos y$, we get $S = \int_{-\pi}^{\pi}\int_{-3}^{3}\sqrt{\sin^2 y + x^2\cos^2 y + 1}\,dx\,dy \approx 62.9714$.



41.



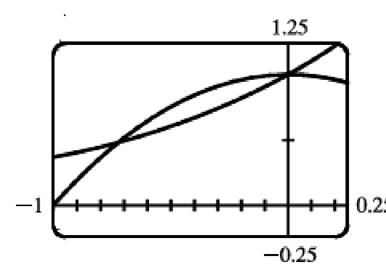
$$\int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(x^3+xy^2)\,dy\,dx = \int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}x(x^2+y^2)\,dy\,dx$$
$$= \int_{-\pi/2}^{\pi/2}\int_0^3(r\cos\theta)(r^2)\,r\,dr\,d\theta$$
$$= \int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta\int_0^3 r^4\,dr$$
$$= \left[\sin\theta\right]_{-\pi/2}^{\pi/2}\left[\tfrac{1}{5}r^5\right]_0^3 = 2\cdot\tfrac{1}{5}(243) = \tfrac{486}{5} = 97.2$$

42. The region of integration is the solid hemisphere $x^2+y^2+z^2 \le 4$, $x \ge 0$.

$$\int_{-2}^{2}\int_0^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$$
$$= \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^2(\rho\sin\phi\sin\theta)^2\left(\sqrt{\rho^2}\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_{-\pi/2}^{\pi/2}\sin^2\theta\,d\theta\int_0^{\pi}\sin^3\phi\,d\phi\int_0^2\rho^5\,d\rho$$
$$= \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2}\left[-\tfrac{1}{3}(2+\sin^2\phi)\cos\phi\right]_0^{\pi}\left[\tfrac{1}{6}\rho^6\right]_0^2 = \left(\tfrac{\pi}{2}\right)\left(\tfrac{2}{3}+\tfrac{2}{3}\right)\left(\tfrac{32}{3}\right) = \tfrac{64}{9}\pi$$

43. From the graph, it appears that $1 - x^2 = e^x$ at $x \approx -0.71$ and at $x = 0$, with $1 - x^2 > e^x$ on $(-0.71, 0)$. So the desired integral is

$$\iint_D y^2\,dA \approx \int_{-0.71}^{0}\int_{e^x}^{1-x^2} y^2\,dy\,dx$$
$$= \tfrac{1}{3}\int_{-0.71}^{0}\left[(1-x^2)^3 - e^{3x}\right]dx$$
$$= \tfrac{1}{3}\left[x - x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7 - \tfrac{1}{3}e^{3x}\right]_{-0.71}^{0} \approx 0.0512$$



44. Let the tetrahedron be called T. The front face of T is given by the plane $x + \tfrac{1}{2}y + \tfrac{1}{3}z = 1$, or $z = 3 - 3x - \tfrac{3}{2}y$, which intersects the xy-plane in the line $y = 2 - 2x$. So the total mass is

$m = \iiint_T \rho(x,y,z)\,dV = \int_0^1\int_0^{2-2x}\int_0^{3-3x-3y/2}(x^2+y^2+z^2)\,dz\,dy\,dx = \tfrac{7}{5}$. The center of mass is

$(\overline{x},\overline{y},\overline{z}) = \left(m^{-1}\iiint_T x\rho(x,y,z)\,dV, m^{-1}\iiint_T y\rho(x,y,z)\,dV, m^{-1}\iiint_T z\rho(x,y,z)\,dV\right) = \left(\tfrac{4}{21},\tfrac{11}{21},\tfrac{8}{7}\right)$.

45. (a) $f(x,y)$ is a joint density function, so we know that $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Since $f(x,y) = 0$ outside the rectangle $[0,3]\times[0,2]$, we can say

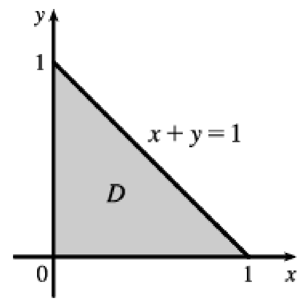
$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^3\int_0^2 C(x+y)\,dy\,dx$$
$$= C\int_0^3 \left[xy + \tfrac{1}{2}y^2\right]_{y=0}^{y=2} dx = C\int_0^3 (2x+2)\,dx = C\left[x^2+2x\right]_0^3 = 15C$$

Then $15C = 1 \quad\Rightarrow\quad C = \frac{1}{15}$.

(b) $P(X\le 2, Y\ge 1) = \int_{-\infty}^{2}\int_1^{\infty} f(x,y)\,dy\,dx = \int_0^2\int_1^2 \frac{1}{15}(x,y)\,dy\,dx = \frac{1}{15}\int_0^2 \left[xy+\frac{1}{2}y^2\right]_{y=1}^{y=2} dx$

$$= \tfrac{1}{15}\int_0^2 \left(x+\tfrac{3}{2}\right)dx = \tfrac{1}{15}\left[\tfrac{1}{2}x^2+\tfrac{3}{2}x\right]_0^2 = \tfrac{1}{3}$$

(c) $P(X+Y\le 1) = P((X,Y)\in D)$ where D is the triangular region shown in the figure. Thus



$$P(X+Y\le 1) = \iint_D f(x,y)\,dA = \int_0^1\int_0^{1-x} \tfrac{1}{15}(x+y)\,dy\,dx$$
$$= \tfrac{1}{15}\int_0^1 \left[xy+\tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx$$
$$= \tfrac{1}{15}\int_0^1 \left[x(1-x)+\tfrac{1}{2}(1-x)^2\right]dx$$
$$= \tfrac{1}{30}\int_0^1 (1-x^2)\,dx = \tfrac{1}{30}\left[x-\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{45}$$

46.
Each lamp has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{800} e^{-t/800} & \text{if } t \geq 0 \end{cases}$$

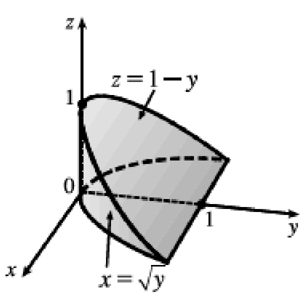
If X, Y, and Z are the lifetimes of the individual bulbs, then X, Y, and Z are independent, so the joint density function is the product of the individual density functions:

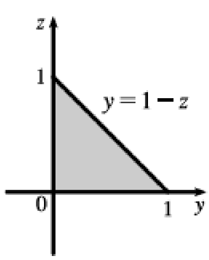
$$f(x,y,z) = \begin{cases} \frac{1}{800^3} e^{-(x+y+z)/800} & \text{if } x \geq 0, y \geq 0, z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that all three bulbs fail within a total of 1000 hours is $P(X+Y+Z \leq 1000)$, or equivalently $P((X,Y,Z) \in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane $x+y+z=1000$. The plane $x+y+z=1000$ meets the xy-plane in the line $x+y=1000$, so we have

$$\begin{aligned} P(X+Y+Z \leq 1000) &= \iiint_E f(x,y,z)\,dV = \int_0^{1000}\int_0^{1000-x}\int_0^{1000-x-y} \tfrac{1}{800^3} e^{-(x+y+z)/800}\,dz\,dy\,dx \\ &= \tfrac{1}{800^3}\int_0^{1000}\int_0^{1000-x} -800\left[e^{-(x+y+z)/800}\right]_{z=0}^{z=1000-x-y} dy\,dx \\ &= \tfrac{-1}{800^2}\int_0^{1000}\int_0^{1000-x}\left[e^{-5/4} - e^{-(x+y)/800}\right] dy\,dx \\ &= \tfrac{-1}{800^2}\int_0^{1000}\left[e^{-5/4}y + 800e^{-(x+y)/800}\right]_{y=0}^{y=1000-x} dx \\ &= \tfrac{-1}{800^2}\int_0^{1000}\left[e^{-5/4}(1800-x) - 800e^{-x/800}\right] dx \\ &= \tfrac{-1}{800^2}\left[-\tfrac{1}{2}e^{-5/4}(1800-x)^2 + 800^2 e^{-x/800}\right]_0^{1000} \\ &= \tfrac{-1}{800^2}\left[-\tfrac{1}{2}e^{-5/4}(800)^2 + 800^2 e^{-5/4} + \tfrac{1}{2}e^{-5/4}(1800)^2 - 800^2\right] \\ &= 1 - \tfrac{97}{32}e^{-5/4} \approx 0.1315 \end{aligned}$$

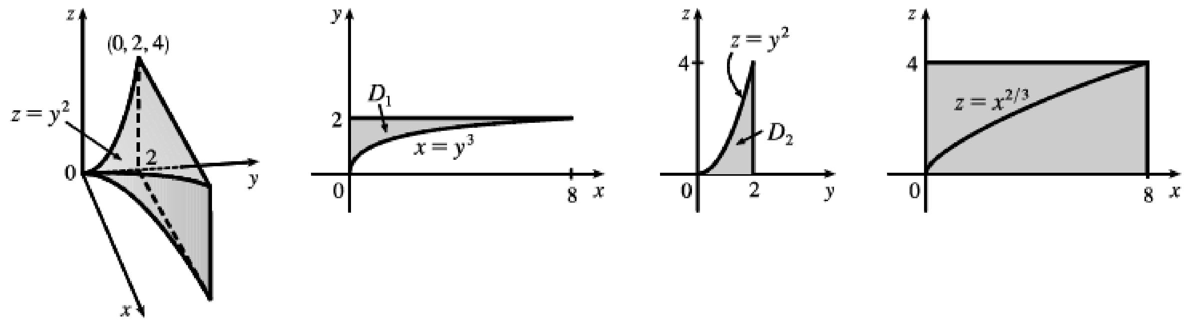
47.





$$\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz$$

48.



$\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x,y,z)\, dz\, dx\, dy = \iiint_E f(x,y,z)\, dV$ where $E = \{(x,y,z) \mid 0 \le y \le 2, 0 \le x \le y^3, 0 \le z \le y^2\}$.

If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$D_1 = \{(x,y) \mid 0 \le y \le 2, 0 \le x \le y^3\} = \{(x,y) \mid 0 \le x \le 8, \sqrt[3]{x} \le y \le 2\}$,

$D_2 = \{(y,z) \mid 0 \le z \le 4, \sqrt{z} \le y \le 2\} = \{(y,z) \mid 0 \le y \le 2, 0 \le z \le y^2\}$, $D_3 = \{(x,z) \mid 0 \le x \le 8, 0 \le z \le 4\}$.

Therefore we have

$$\begin{aligned}\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x,y,z)\, dz\, dx\, dy &= \int_0^8 \int_{\sqrt[3]{x}}^2 \int_0^{y^2} f(x,y,z)\, dz\, dy\, dx = \int_0^4 \int_{\sqrt{z}}^2 \int_0^{y^3} f(x,y,z)\, dx\, dy\, dz \\ &= \int_0^2 \int_0^{y^2} \int_0^{y^3} f(x,y,z)\, dx\, dz\, dy \\ &= \int_0^8 \int_0^{x^{2/3}} \int_{\sqrt[3]{x}}^2 f(x,y,z)\, dy\, dz\, dx + \int_0^8 \int_{x^{2/3}}^4 \int_{\sqrt{z}}^2 f(x,y,z)\, dy\, dz\, dx \\ &= \int_0^4 \int_0^{z^{3/2}} \int_{\sqrt{z}}^2 f(x,y,z)\, dy\, dx\, dz + \int_0^4 \int_{z^{3/2}}^8 \int_{\sqrt[3]{x}}^2 f(x,y,z)\, dy\, dx\, dz\end{aligned}$$

49. Since $u = x - y$ and $v = x + y$, $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(v-u)$.

Thus $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$ and $\displaystyle\iint_R \frac{x-y}{x+y}\, dA = \int_2^4 \int_{-2}^0 \frac{u}{v}\left(\frac{1}{2}\right) du\, dv = -\int_2^4 \frac{dv}{v} = -\ln 2$.

50. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = 8uvw$, so

$$\begin{aligned}V &= \iiint_E dV = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 8uvw\, dw\, dv\, du = \int_0^1 \int_0^{1-u} 4uv(1-u-v)^2\, du \\ &= \int_0^1 \int_0^{1-u} \left[4u(1-u)^2 v - 8u(1-u)v^2 + 4uv^3\right] dv\, du \\ &= \int_0^1 \left[2u(1-u)^4 - \tfrac{8}{3}u(1-u)^4 + u(1-u)^4\right] du = \int_0^1 \tfrac{1}{3}u(1-u)^4 du \\ &= \int_0^1 \tfrac{1}{3}\left[(1-u)^4 - (1-u)^5\right] du = \tfrac{1}{3}\left[-\tfrac{1}{5}(1-u)^5 + \tfrac{1}{6}(1-u)^6\right]_0^1 = \tfrac{1}{3}\left(-\tfrac{1}{6} + \tfrac{1}{5}\right) = \tfrac{1}{90}\end{aligned}$$

51. Let $u = y - x$ and $v = y + x$ so $x = y - u = (v - x) - u \quad \Rightarrow \quad x = \frac{1}{2}(v - u)$ and $y = v - \frac{1}{2}(v - u) = \frac{1}{2}(v + u)$.

$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| = \left|-\frac{1}{2}\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right)\right| = \left|-\frac{1}{2}\right| = \frac{1}{2}$. R is the image under this transformation of the square with vertices $(u, v) = (0, 0)$, $(-2, 0)$, $(0, 2)$, and $(-2, 2)$. So

$$\iint_R xy\, dA = \int_0^2 \int_{-2}^0 \frac{v^2 - u^2}{4}\left(\frac{1}{2}\right) du\, dv = \frac{1}{8}\int_0^2 \left[v^2 u - \frac{1}{3}u^3\right]_{u=-2}^{u=0} dv = \frac{1}{8}\int_0^2 \left(2v^2 - \frac{8}{3}\right) dv = \frac{1}{8}\left[\frac{2}{3}v^3 - \frac{8}{3}v\right]_0^2 = 0$$

This result could have been anticipated by symmetry, since the integrand is an odd function of y and R is symmetric about the x-axis.

52.
By the Extreme Value Theorem (14.7.8), f has an absolute minimum value m and an absolute maximum value M in D. Then by Property 15.3.11, $mA(D) \le \iint_D f(x, y)\, dA \le MA(D)$. Dividing through by the positive number $A(D)$, we get $m \le \frac{1}{A(D)} \iint_D f(x, y)\, dA \le M$. This says that the average value of f over D lies between m and M. But f is continuous on D and takes on the values m and M, and so by the Intermediate Value Theorem must take on all values between m and M. Specifically, there exists a point (x_0, y_0) in D such that $f(x_0, y_0) = \frac{1}{A(D)} \iint_D f(x, y)\, dA$ or equivalently $\iint_D f(x, y)\, dA = f(x_0, y_0)\, A(D)$.

53. For each r such that D_r lies within the domain, $A(D_r) = \pi r^2$, and by the Mean Value Theorem for Double Integrals there exists (x_r, y_r) in D_r such that $f(x_r, y_r) = \frac{1}{\pi r^2} \iint_{D_r} f(x, y)\, dA$. But $\lim_{r \to 0^+} (x_r, y_r) = (a, b)$, so $\lim_{r \to 0^+} \frac{1}{\pi r^2} \iint_{D_r} f(x, y)\, dA = \lim_{r \to 0^+} f(x_r, y_r) = f(a, b)$ by the continuity of f.

54. (a) $\iint_D \frac{1}{(x^2+y^2)^{n/2}}\,dA = \int_0^{2\pi}\int_r^R \frac{1}{(t^2)^{n/2}}t\,dt\,d\theta = 2\pi\int_r^R t^{1-n}\,dt$

$$= \begin{cases} \left.\frac{2\pi}{2-n}t^{2-n}\right]_r^R = \frac{2\pi}{2-n}\left(R^{2-n}-r^{2-n}\right) & \text{if } n \neq 2 \\ 2\pi\ln(R/r) & \text{if } n = 2 \end{cases}$$

(b) The integral in part (a) has a limit as $r \to 0^+$ for all values of n such that $2-n>0 \quad\Leftrightarrow\quad n<2$.

(c) $\iiint_E \frac{1}{(x^2+y^2+z^2)^{n/2}}\,dV = \int_r^R\int_0^{\pi}\int_0^{2\pi} \frac{1}{(\rho^2)^{n/2}}\rho^2\sin\phi\,d\theta\,d\phi\,d\rho = 2\pi\int_r^R\int_0^{\pi}\rho^{2-n}\sin\phi\,d\phi\,d\rho$

$$= \begin{cases} \left.\frac{4\pi}{3-n}\rho^{3-n}\right]_r^R = \frac{4\pi}{3-n}\left(R^{3-n}-r^{3-n}\right) & \text{if } n \neq 3 \\ 4\pi\ln(R/r) & \text{if } n = 3 \end{cases}$$

(d) As $r \to 0^+$, the above integral has a limit, provided that $3-n>0 \quad\Leftrightarrow\quad n<3$.