1. See Definitions 1 and 2 in Section 16.1. A vector field can represent, for example, the wind velocity at any location in space, the speed and direction of the ocean current at any location, or the force vectors of Earth's gravitational field at a location in space.

2. (a) A conservative vector field $\mathbf{F}$ is a vector field which is the gradient of some scalar function f.

   (b) The function f in part (a) is called a potential function for $\mathbf{F}$, that is, $\mathbf{F} = \nabla f$.

3. (a) See Definition 16.2.2.

   (b) We normally evaluate the line integral using Formula 16.2.3.

   (c) The mass is $m = \int_C \rho(x, y)\ ds$, and the center of mass is $(\overline{x}, \overline{y})$ where $\overline{x} = \frac{1}{m} \int_C x\rho(x, y)\ ds$, $\overline{y} = \frac{1}{m} \int_C y\rho(x, y)\ ds$.

   (d) See (5) and (6) in Section 16.2 for plane curves; we have similar definitions when C is a space curve [see the equation preceding (10) in Section 16.2].

   (e) For plane curves, see Equations 16.2.7. We have similar results for space curves [see the equation preceding (10) in Section 16.2].

4. (a) See Definition 16.2.13.

   (b) If $\mathbf{F}$ is a force field, $\int_C \mathbf{F} \cdot d\mathbf{r}$ represents the work done by $\mathbf{F}$ in moving a particle along the curve C.

   (c) $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$

5. See Theorem 16.3.2.

6. (a) $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if the line integral has the same value for any two curves that have the same initial and terminal points.

   (b) See Theorem 16.3.4.

7. See the statement of Green's Theorem on page 1108 [ET 1084].

8. See Equations 16.4.5.

9. (a) $\operatorname{curl} \mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = \nabla \times \mathbf{F}$

   (b) $\operatorname{div} \mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = \nabla \cdot \mathbf{F}$

   (c) For curl $\mathbf{F}$, see the discussion accompanying Figure 1 on page 1118 [ET 1094] as well as Figure 6 and the accompanying discussion on page 1150 [ET 1126]. For div $\mathbf{F}$, see the discussion following Example 5 on page 1119 [ET 1095] as well as the discussion preceding (8) on page 1157 [ET 1133].

10. See Theorem 16.3.6; see Theorem 16.5.4.

11. (a) See (1) and (2) and the accompanying discussion in Section 16.6; See Figure 4 and the accompanying discussion on page 1124 [ET 1100].

(b) See Definition 16.6.6.

(c) See Equation 16.6.9.

12. (a) See (1) in Section 16.7.

(b) We normally evaluate the surface integral using Formula 16.7.2.

(c) See Formula 16.7.4.

(d) The mass is $m = \iint_S \rho(x, y, z)\, dS$ and the center of mass is $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = \frac{1}{m}\iint_S x\rho(x, y, z)\, dS$, $\overline{y} = \frac{1}{m}\iint_S y\rho(x, y, z)\, dS$, $\overline{z} = \frac{1}{m}\iint_S z\rho(x, y, z)\, dS$.

13. (a) See Figures 6 and 7 and the accompanying discussion in Section 16.7. A Möbius strip is a nonorientable surface; see Figures 4 and 5 and the accompanying discussion on page 1139 [ET 1115].

(b) See Definition 16.7.8.

(c) See Formula 16.7.9.

(d) See Formula 16.7.10.

14. See the statement of Stokes' Theorem on page 1146 [ET 1122].

15. See the statement of the Divergence Theorem on page 1153 [ET 1129].

16. In each theorem, we have an integral of a "derivative" over a region on the left side, while the right side involves the values of the original function only on the boundary of the region.

1. False; $\operatorname{div} \mathbf{F}$ is a scalar field.

2. True. (See Definition 16.5.1.)

3. True, by Theorem 16.5.3 and the fact that $\operatorname{div} \mathbf{0} = 0$.

4. True, by Theorem 16.3.2.

5. False. See Exercise 16.3.35. (But the assertion is true if D is simply-connected; see Theorem 16.3.6.)

6. False. See the discussion accompanying Figure 8 on page 1092 [ET 1068].

7. False. For example, $\operatorname{div}(y\,\mathbf{i}) = \mathbf{0} = \operatorname{div}(x\,\mathbf{j})$ but $y\,\mathbf{i} \neq x\,\mathbf{j}$.

8. True. Line integrals of conservative vector fields are independent of path, and by Theorem 16.3.3, work $= \int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path C.

9. True. See Exercise 16.5.24.

10. False. $\mathbf{F} \cdot \mathbf{G}$ is a scalar field, so $\operatorname{curl}(\mathbf{F} \cdot \mathbf{G})$ has no meaning.

11. True. Apply the Divergence Theorem and use the fact that $\operatorname{div} \mathbf{F} = 0$.

12. False by Theorem 16.5.11, because if it were true, then $\operatorname{div} \operatorname{curl} \mathbf{F} = 3 \neq 0$.

2. We can parametrize C by $x = x$, $y = x^2$, $0 \leq x \leq 1$ so

$$\int_C x\,ds = \int_0^1 x\sqrt{1 + (2x)^2}\,dx = \tfrac{1}{12}(1 + 4x^2)^{3/2}\Big]_0^1 = \tfrac{1}{12}\left(5\sqrt{5} - 1\right).$$

4. $x = 3\cos t \quad \Rightarrow \quad dx = -3\sin t\,dt$, $y = 2\sin t \quad \Rightarrow \quad dy = 2\cos t\,dt$, $0 \leq t \leq 2\pi$, so

$$\begin{aligned}\int_C y\,dx + \left(x + y^2\right)dy &= \int_0^{2\pi} \left[(2\sin t)(-3\sin t) + (3\cos t + 4\sin^2 t)(2\cos t)\right] dt \\ &= \int_0^{2\pi} (-6\sin^2 t + 6\cos^2 t + 8\sin^2 t\,\cos t)\,dt = \int_0^{2\pi} \left[6(\cos^2 t - \sin^2 t) + 8\sin^2 t\,\cos t\right] dt \\ &= \int_0^{2\pi} (6\cos 2t + 8\sin^2 t\,\cos t)\,dt = 3\sin 2t + \tfrac{8}{3}\sin^3 t\Big]_0^{2\pi} = 0\end{aligned}$$

Or: Notice that $\frac{\partial}{\partial y}(y) = 1 = \frac{\partial}{\partial x}\left(x + y^2\right)$, so $\mathbf{F}(x, y) = \left\langle y, x + y^2\right\rangle$ is a conservative vector field. Since C is a closed curve, $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y\,dx + (x + y^2)\,dy = 0$.

6. $$\begin{aligned}\int_C \sqrt{xy}\,dx + e^y\,dy + xz\,dz &= \int_0^1 \left(\sqrt{t^4 \cdot t^2} \cdot 4t^3 + e^{t^2} \cdot 2t + t^4 \cdot t^3 \cdot 3t^2\right) dt = \int_0^1 (4t^6 + 2te^{t^2} + 3t^9)\,dt \\ &= \left[\tfrac{4}{7}t^7 + e^{t^2} + \tfrac{3}{10}t^{10}\right]_0^1 = e - \tfrac{9}{70}\end{aligned}$$

8. $\mathbf{F}(\mathbf{r}(t)) = (\sin t)(1 + t)\,\mathbf{i} + (\sin^2 t)\,\mathbf{j}$, $\mathbf{r}'(t) = \cos t\,\mathbf{i} + \mathbf{j}$ and

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi ((1 + t)\sin t\cos t + \sin^2 t)\,dt = \int_0^\pi \left(\tfrac{1}{2}(1 + t)\sin 2t + \sin^2 t\right) dt \\ &= \left[\tfrac{1}{2}\left((1 + t)\left(-\tfrac{1}{2}\cos 2t\right) + \tfrac{1}{4}\sin 2t\right) + \tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right]_0^\pi = \tfrac{\pi}{4}\end{aligned}$$

10. (a) C: $x = 3 - 3t$, $y = \frac{\pi}{2}t$, $z = 3t$, $0 \leq t \leq 1$. Then

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left[3t\,\mathbf{i} + (3 - 3t)\,\mathbf{j} + \tfrac{\pi}{2}t\,\mathbf{k}\right] \cdot \left[-3\,\mathbf{i} + \tfrac{\pi}{2}\,\mathbf{j} + 3\,\mathbf{k}\right] dt = \int_0^1 \left[-9t + \tfrac{3\pi}{2}\right] dt = \tfrac{1}{2}(3\pi - 9).$$

(b) $$\begin{aligned}W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + t\,\mathbf{k}) \cdot (-3\sin t\,\mathbf{i} + \mathbf{j} + 3\cos t\,\mathbf{k})\,dt \\ &= \int_0^{\pi/2} (-9\sin^2 t + 3\cos t + 3t\cos t)\,dt = \left[-\tfrac{9}{2}(t - \sin t\cos t) + 3\sin t + 3(t\sin t + \cos t)\right]_0^{\pi/2} \\ &= -\tfrac{9\pi}{4} + 3 + \tfrac{3\pi}{2} - 3 = -\tfrac{3\pi}{4}\end{aligned}$$

12.
$\mathbf{F}$ is defined on all of $\mathbb{R}^3$, its components have continuous partial derivatives, and

curl $\mathbf{F} = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (\cos y - \cos y)\,\mathbf{k} = \mathbf{0}$, so $\mathbf{F}$ is conservative by Theorem 16.5.4. Thus there exists a function

f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = \sin y$ implies $f(x,y,z) = x \sin y + g(y,z)$ and then

$f_y(x,y,z) = x\cos y + g_y(y,z)$. But $f_y(x,y,z) = x\cos y$, so $g_y(y,z) = 0 \quad \Rightarrow \quad g(y,z) = h(z)$. Then

$f(x,y,z) = x\sin y + h(z)$ implies $f_z(x,y,z) = h'(z)$. But $f_z(x,y,z) = -\sin z$, so $h(z) = \cos z + K$. Thus a potential

function for $\mathbf{F}$ is $f(x,y,z) = x\sin y + \cos z + K$.

14. Here curl $\mathbf{F} = \mathbf{0}$, the domain of $\mathbf{F}$ is $\mathbb{R}^3$, and the components of $\mathbf{F}$ have continuous partial derivatives, so $\mathbf{F}$ is conservative.

Furthermore $f(x,y,z) = xe^y + ye^z$ is a potential function for $\mathbf{F}$. Then $\int_C \mathbf{F}\cdot d\mathbf{r} = f(4,0,3) - f(0,2,0) = 4 - 2 = 2$.

16. $\int_C \sqrt{1+x^3}\,dx + 2xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}\left(\sqrt{1+x^3}\right)\right] dA = \int_0^1 \int_0^{3x} (2y - 0)\,dy\,dx = \int_0^1 9x^2\,dx = 3x^3\Big]_0^1 = 3$

18. curl $\mathbf{F} = (0 - e^{-y}\cos z)\,\mathbf{i} - (e^{-z}\cos x - 0)\,\mathbf{j} + (0 - e^{-x}\cos y)\,\mathbf{k} = -e^{-y}\cos z\,\mathbf{i} - e^{-z}\cos x\,\mathbf{j} - e^{-x}\cos y\,\mathbf{k}$,

div $\mathbf{F} = -e^{-x}\sin y - e^{-y}\sin z - e^{-z}\sin x$

20.
Let $\mathbf{F} = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G} = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$ be vector fields whose first partials exist and are continuous. Then

$$\begin{aligned}\mathbf{F}\,\mathrm{div}\,\mathbf{G} - \mathbf{G}\,\mathrm{div}\,\mathbf{F} = &\left[P_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{i} + Q_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{j} + R_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{k}\right] \\ &- \left[P_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{i} + Q_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{j}\right. \\ &\left. + R_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{k}\right]\end{aligned}$$

and

$$\begin{aligned}(\mathbf{G}\cdot\nabla)\,\mathbf{F} - (\mathbf{F}\cdot\nabla)\,\mathbf{G} = &\left[\left(P_2\,\frac{\partial P_1}{\partial x} + Q_2\,\frac{\partial P_1}{\partial y} + R_2\,\frac{\partial P_1}{\partial z}\right)\mathbf{i} + \left(P_2\,\frac{\partial Q_1}{\partial x} + Q_2\,\frac{\partial Q_1}{\partial y} + R_2\,\frac{\partial Q_1}{\partial z}\right)\mathbf{j}\right. \\ &\left. + \left(P_2\,\frac{\partial R_1}{\partial x} + Q_2\,\frac{\partial R_1}{\partial y} + R_2\,\frac{\partial R_1}{\partial z}\right)\mathbf{k}\right] \\ &- \left[\left(P_1\,\frac{\partial P_2}{\partial x} + Q_1\,\frac{\partial P_2}{\partial y} + R_1\,\frac{\partial P_2}{\partial z}\right)\mathbf{i} + \left(P_1\,\frac{\partial Q_2}{\partial x} + Q_1\,\frac{\partial Q_2}{\partial y} + R_1\,\frac{\partial Q_2}{\partial z}\right)\mathbf{j}\right. \\ &\left. + \left(P_1\,\frac{\partial R_2}{\partial x} + Q_1\,\frac{\partial R_2}{\partial y} + R_1\,\frac{\partial R_2}{\partial z}\right)\mathbf{k}\right]\end{aligned}$$

Hence

$$\mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} + (\mathbf{G}\cdot\nabla)\,\mathbf{F} - (\mathbf{F}\cdot\nabla)\,\mathbf{G}$$

$$\begin{aligned}
&= \left[\left(P_1\,\frac{\partial Q_2}{\partial y} + Q_2\,\frac{\partial P_1}{\partial x}\right) - \left(P_2\,\frac{\partial Q_1}{\partial y} + Q_1\,\frac{\partial P_2}{\partial y}\right)\right.\\
&\qquad\left. - \left(P_2\,\frac{\partial R_1}{\partial z} + R_1\,\frac{\partial P_2}{\partial z}\right) + \left(P_1\,\frac{\partial R_2}{\partial z} + R_2\,\frac{\partial P_1}{\partial z}\right)\right]\mathbf{i}\\
&\quad + \left[\left(Q_1\,\frac{\partial R_2}{\partial z} + R_2\,\frac{\partial Q_1}{\partial z}\right) - \left(Q_2\,\frac{\partial R_1}{\partial z} + R_1\,\frac{\partial Q_2}{\partial z}\right)\right.\\
&\qquad\left. - \left(P_1\,\frac{\partial Q_2}{\partial x} + Q_2\,\frac{\partial P_1}{\partial x}\right) + \left(P_2\,\frac{\partial Q_1}{\partial x} + Q_1\,\frac{\partial P_2}{\partial x}\right)\right]\mathbf{j}\\
&\quad + \left[\left(P_2\,\frac{\partial R_1}{\partial x} + R_1\,\frac{\partial P_2}{\partial x}\right) - \left(P_1\,\frac{\partial R_2}{\partial x} + R_2\,\frac{\partial P_1}{\partial x}\right)\right.\\
&\qquad\left. - \left(Q_1\,\frac{\partial R_2}{\partial y} + R_2\,\frac{\partial Q_1}{\partial y}\right) + \left(Q_2\,\frac{\partial R_1}{\partial y} + R_1\,\frac{\partial Q_2}{\partial y}\right)\right]\mathbf{k}\\
&= \left[\frac{\partial}{\partial y}\,(P_1Q_2 - P_2Q_1) - \frac{\partial}{\partial z}\,(P_2R_1 - P_1R_2)\right]\mathbf{i}\\
&\quad + \left[\frac{\partial}{\partial z}\,(Q_1R_2 - Q_2R_1) - \frac{\partial}{\partial x}\,(P_1Q_2 - P_2Q_1)\right]\mathbf{j}\\
&\quad + \left[\frac{\partial}{\partial x}\,(P_2R_1 - P_1R_2) - \frac{\partial}{\partial y}\,(Q_1R_2 - Q_2R_1)\right]\mathbf{k}\\
&= \operatorname{curl}(\mathbf{F}\times\mathbf{G})
\end{aligned}$$

22. $\nabla^2(fg) = \dfrac{\partial^2(fg)}{\partial x^2} + \dfrac{\partial^2(fg)}{\partial y^2} + \dfrac{\partial^2(fg)}{\partial z^2}$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\,g + f\,\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\,g + f\,\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\,g + f\,\frac{\partial g}{\partial z}\right) \quad \text{[Product Rule]}$$

$$= \frac{\partial^2 f}{\partial x^2}\,g + 2\,\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\,\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\,g + 2\,\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}$$

$$+ f\,\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\,g + 2\,\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\,\frac{\partial^2 g}{\partial z^2} \quad \text{[Product Rule]}$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + g\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + 2\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle \cdot \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right\rangle$$

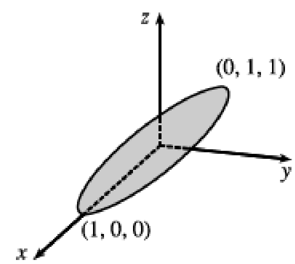
$$= f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$$

Another method: Using the rules in Exercises 14.6.37(b) and 16.5.25, we have

$$\nabla^2(fg) = \nabla \cdot \nabla(fg) = \nabla \cdot (g\,\nabla f + f\,\nabla g) = \nabla g \cdot \nabla f + g\nabla \cdot \nabla f + \nabla f \cdot \nabla g + f\nabla \cdot \nabla g$$

$$= g\,\nabla^2 f + f\,\nabla^2 g + 2\nabla f \cdot \nabla g$$

24. (a) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so C lies on the circular cylinder $x^2 + y^2 = 1$. But also $y = z$, so C lies on the plane $y = z$. Thus C is the intersection of the plane $y = z$ and the cylinder $x^2 + y^2 = 1$.



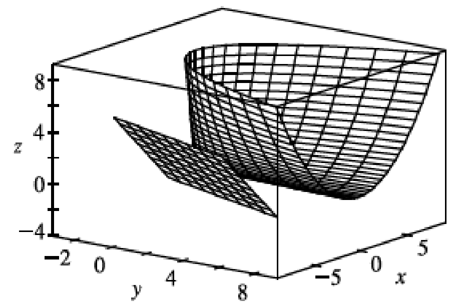
(b) Apply Stokes' Theorem, $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$:

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xe^{2y} & 2x^2e^{2y} + 2y\cot z & -y^2\csc^2 z \end{vmatrix} = \left\langle -2y\csc^2 z - (-2y\csc^2 z), 0, 4xe^{2y} - 4xe^{2y}\right\rangle = \mathbf{0}$$

Therefore $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$.

26. (a) $\mathbf{r}_u = -v\,\mathbf{j} + 2u\,\mathbf{k}$, $\mathbf{r}_v = 2v\,\mathbf{i} - u\,\mathbf{j}$ and $\mathbf{r}_u \times \mathbf{r}_v = 2u^2\,\mathbf{i} + 4uv\,\mathbf{j} + 2v^2\,\mathbf{k}$. Since the point $(4, -2, 1)$ corresponds to $u = 1$, $v = 2$ (or $u = -1$, $v = -2$ but $\mathbf{r}_u \times \mathbf{r}_v$ is the same for both), a normal vector to the surface at $(4, -2, 1)$ is $2\,\mathbf{i} + 8\,\mathbf{j} + 8\,\mathbf{k}$ and an equation of the tangent plane is $2x + 8y + 8z = 0$ or $x + 4y + 4z = 0$.

(b)



(c) By Definition 16.6.6, the area of S is given by

$$A(S) = \int_0^3 \int_{-3}^3 \sqrt{(2u^2)^2 + (4uv)^2 + (2v^2)^2}\,dv\,du = 2\int_0^3 \int_{-3}^3 \sqrt{u^4 + 4u^2v^2 + v^4}\,dv\,du.$$

(d) By Equation 16.7.9, the surface integral is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^3 \int_{-3}^3 \left\langle \frac{(u^2)^2}{1 + (v^2)^2}, \frac{(v^2)^2}{1 + (-uv)^2}, \frac{(-uv)^2}{1 + (u^2)^2} \right\rangle \cdot \langle 2u^2, 4uv, 2v^2 \rangle\,dv\,du$$

$$= \int_0^3 \int_{-3}^3 \left(\frac{2u^6}{1 + v^4} + \frac{4uv^5}{1 + u^2v^2} + \frac{2u^2v^4}{1 + u^4} \right) dv\,du \approx 1524.0190$$

28. $z = f(x, y) = 4 + x + y$ with $0 \le x^2 + y^2 \le 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -\mathbf{i} - \mathbf{j} + \mathbf{k}$. Then

$$\iint_S (x^2 z + y^2 z)\,dS = \iint_{x^2 + y^2 \le 4} (x^2 + y^2)(4 + x + y)\sqrt{3}\,dA$$

$$= \int_0^2 \int_0^{2\pi} \sqrt{3}\,r^3 (4 + r\cos\theta + r\sin\theta)\,d\theta\,dr = \int_0^2 8\pi\sqrt{3}\,r^3\,dr = 32\pi\sqrt{3}$$

30. $z = f(x, y) = x^2 + y^2$, $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (because of upward orientation) and $\mathbf{F}(\mathbf{r}(x, y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) = -2x^3 - 2xy^2 + x^2 + y^2$. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2 + y^2 \le 1} (-2x^3 - 2xy^2 + x^2 + y^2)\,dA$$

$$= \int_0^1 \int_0^{2\pi} (-2r^3\cos^3\theta - 2r^3\cos\theta\,\sin^2\theta + r^2)\,r\,dr\,d\theta = \int_0^1 r^3 (2\pi)\,dr = \tfrac{\pi}{2}$$

32. $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$ where C: $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$, so $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$, $\mathbf{F}(\mathbf{r}(t)) = 8\cos^2 t\,\sin t\,\mathbf{i} + 2\sin t\,\mathbf{j} + e^{4\cos t\sin t}\,\mathbf{k}$, and $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -16\cos^2 t\sin^2 t + 4\sin t\cos t$. Thus

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-16\cos^2 t\,\sin^2 t + 4\sin t\,\cos t)\,dt = \left[-16\left(-\tfrac{1}{4}\sin t\,\cos^3 t + \tfrac{1}{16}\sin 2t + \tfrac{1}{8}t\right) + 2\sin^2 t\right]_0^{2\pi} = -4\pi.$$

34. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 3(x^2 + y^2 + z^2)\,dV = \int_0^{2\pi} \int_0^1 \int_0^2 (3r^2 + 3z^2)\,r\,dz\,dr\,d\theta = 2\pi\int_0^1 (6r^3 + 8r)\,dr = 11\pi$

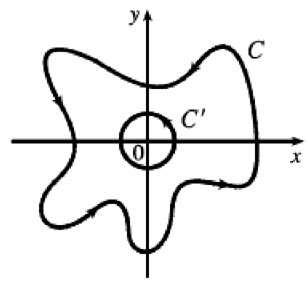
36.
Here we must use Equation 16.9.7 since $\mathbf{F}$ is not defined at the origin. Let S_1 be the sphere of radius 1 with center at the origin and outer unit normal $\mathbf{n}_1$. Let S_2 be the surface of the ellipsoid with outer unit normal $\mathbf{n}_2$ and let E be the solid region between S_1 and S_2. Then the outward flux of $\mathbf{F}$ through the ellipsoid is given by

$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 \, dS = -\iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1) \, dS + \iiint_E \operatorname{div} \mathbf{F} \, dV$. But $\mathbf{F} = \mathbf{r}/\left|\mathbf{r}\right|^3$, so

$\operatorname{div} \mathbf{F} = \nabla \cdot \left(\left|\mathbf{r}\right|^{-3} \mathbf{r}\right) = \left|\mathbf{r}\right|^{-3}(\nabla \cdot \mathbf{r}) + \mathbf{r} \cdot \left(\nabla \left|\mathbf{r}\right|^{-3}\right) = \left|\mathbf{r}\right|^{-3}(3) + \mathbf{r} \cdot \left(-3\left|\mathbf{r}\right|^{-4}\right)\left(\mathbf{r}\left|\mathbf{r}\right|^{-1}\right) = \mathbf{0}$. [Here we have used Exercises 16.5.30(a) and 16.5.31(a).] And $\mathbf{F} \cdot \mathbf{n}_1 = \dfrac{\mathbf{r}}{\left|\mathbf{r}\right|^3} \cdot \dfrac{\mathbf{r}}{\left|\mathbf{r}\right|} = \left|\mathbf{r}\right|^{-2} = 1$ on S_1.

Thus $\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 \, dS = \iint_{S_1} dS + \iiint_E 0 \, dV = (\text{surface area of the unit sphere}) = 4\pi(1)^2 = 4\pi$.

38. Let C' be the circle with center at the origin and radius a as in the figure. Let D be the region bounded by C and C'. Then D's positively oriented boundary is $C \cup (-C')$. Hence by Green's Theorem



$$\int_C \mathbf{F} \cdot d\mathbf{r} + \int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \mathbf{0}, \text{ so}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot dr = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

$$= \int_0^{2\pi} \left[\frac{2a^3\cos^3 t + 2a^3 \cos t \, \sin^2 t - 2a \sin t}{a^2}(-a \sin t) + \frac{2a^3 \sin^3 t + 2a^3 \cos^2 t \, \sin t + 2a \cos t}{a^2}(a \cos t)\right] dt$$

$$= \int_0^{2\pi} \frac{2a^2}{a^2} \, dt = 4\pi$$

40. The stated conditions allow us to use the Divergence Theorem. Hence $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F}) \, dV = 0$ since $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$.