FINAL FAQ

1. General questions

(1) **Question: How does one parametrize a line or a line segment? For example, how does one parametrize the line segment from the point $(0,1)$ to the point $(1,0)$?**
Answer: To parametrize a line segment you form the velocity vector between the two points. In the example in the question, you form the vector $\mathbf{v} = (1,0) - (0,1) = (1,-1)$. You then multiply this velocity vector by the parameter and add the starting point. In the example, you get:

$$r(t) = (0,1) + t(1,-1) = (t, 1-t), \qquad 0 \leq t \leq 1.$$

The bounds $0 \leq t \leq 1$ mean you only get the line segment between these two points. If you want the whole line then just take $-\infty \leq t \leq \infty$.

(2) **Question: What is the orientation of a surface, and how do you pick a parametrization that matches this orientation?**
Answer: Any parametrization $\mathbf{r}(u,v)$ fixes an orientation: the normal vector for the surface is in the direction of the vector $\mathbf{r}_u \times \mathbf{r}_v$. If you have a parametrization and you want to change the orientation, the easiest way to do this is to swap which variable is the "first" one. For example, consider the parametrizations:

$$\mathbf{r}_1(u,v) = \langle u, v, u-v \rangle,$$
$$\mathbf{r}_2(u,v) = \langle v, u, v-u \rangle.$$

Both $\mathbf{r}_1$ and $\mathbf{r}_2$ parametrize the plane $-x+y+z=0$, butthey have opposite orientations. The only difference between them is that we swapped the roles of u and v. The reason this works is that for any parametrization $\mathbf{r}$, we have $\mathbf{r}_u \times \mathbf{r}_v = -(\mathbf{r}_v \times \mathbf{r}_u)$. Thus, the normals of these vector fields are antiparallel, and thus give opposite orientations to the surface.

(3) **Question: How do you parametrize an ellipsoid?**
Answer: The unit sphere $\left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\right\}$ can be parametrized by:

$$\mathbf{r}(\phi,\theta) = \langle \cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi) \rangle, \qquad 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq 2\pi.$$

It looks like spherical coordinates, but since we're parametrizing a two-dimensional object (a surface), we need exactly two variables. The radial coordinate ρ doesn't appear because we're parametrizing the surface $\rho = 1$.

This parametrization is worth remembering, because it's a useful building block for parametrizing any ellipsoid. An ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, contains the points $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$. In fact, the ellipsoid is just a stretched unit sphere: it stretches by a in the x-direction, by b in the y-direction, and by c in the z-direction. Any stretch like this can be obtained by multiplying the stretch factor in each coordinate:

$$\mathbf{r}(\phi,\theta) = \langle a\cos(\theta)\sin(\phi), b\sin(\theta)\sin(\phi), c\cos(\phi) \rangle.$$

The bounds on ϕ and θ are still the same as the sphere.

(4) **Question: What is the difference between a vector field and a scalar field?**
Answer: For $n \geq 1$, a scalar field is a function $\mathbb{R}^n \to \mathbb{R}$, so it may take in general a vector as input, and then output a scalar. By contrast for $n \geq 2$, a vector field is a function $\mathbb{R}^n \to \mathbb{R}^n$, so it always takes in a vector input, and outputs a vector fo the same dimension.

(5) **Question: What is the difference between change of coordinates and parameterization?**
Answer: Parameterization is the process of using a map to describe a geometric object. Usually we use something we find easy to deal with - a line, circle, rectangle, sphere etc. - and use it to describe something more complicated - a helix, a piece of a plane, a torus, an ellipsoid, etc. So parameterization is formally given by a map $r : \mathbb{R}^m \to \mathbb{R}^n$, where $m \le n$. A change of coordinates is a transformation of the whole space under consideration. You can see it as a special kind of parameterization where $m = n$. So, $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (x + y, x - y)$ is an example of a change of coordinates. In contrast, $r : \mathbb{R} \to \mathbb{R}^2$ given by $r(t) = (t, t^2)$ is a parameterization, but not a change of co-ordinates.

(6) **Question: How can you tell when a vector field is conservative or not?**
Answer: In 2-dimension remember the theorems about conservative vector fields.
(a) On an open connected region D in $\mathbb{R}^2$, a vector field $\mathbf{F}$ is conservative if and only if it is path independent.
(b) On an open connected region D in $\mathbb{R}^2$, a vector field $\mathbf{F}$ is conservative if and only if $\int_C F \cdot dr = 0$ on any closed curve C.
(c) On a simply connected open region D in $\mathbb{R}^2$, a vector field $\mathbf{F} = (P, Q)$ is conservative if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.
To show a vector field is conservative you check any of the three conditions stated above, though (c) is the most commonly used since the first two are difficult in specific examples. Make sure that you check that your domain is simply connected besides checking that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.
To show a vector field is 'not' conservative you may do any of the following:
(a) Find two paths with the same endpoints on which the line integral have different values.
(b) Find a closed curve on which the line integral is not 0.
(c) Show $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$.
Note that an analogue of theorem(c) in 3-dimension tells you that that a vector field $\mathbf{F}$ defined on $\mathbb{R}^3$ is conservative if and only if $\operatorname{curl}(\mathbf{F}) = 0$.

(7) **Question: What do the divergence and curl represent intuitively? How do they relate to each other?**
Answer: The divergence of a vector field measures how much "source" and "sink" the vector field has. Imagine putting a water faucet at the origin and turn it on, the water will flow out away from the origin. The vector field of the flow of water will have positive divergence there, since the faucet is a "source". If you instead add a drain, water will leave through the drain. This is a "sink", and the vector field of the water flow will have negative divergence there. Note that with this intuition, you may interpret a vector field $\mathbf{F}$ with $\operatorname{div}(\mathbf{F}) = 0$ as one with no sources or sinks.
The curl of a vector field corresponds to the rotation of the vector field. The curl is another vector field itself, it's direction is along the axis of rotation and such that the right hand rule is preserved. You can find its direction using the right-hand rule like this: if $\mathbf{F}$ is a vector field that looks like it's rotating around a point, curl your fingers in the direction $\mathbf{F}$ is rotating. Then your thumb will point in the direction of $\operatorname{curl}(\mathbf{F})$.
If $\mathbf{F}$ is any vector field, then we always have $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0$. To verify this just start with a vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$, compute the curl and then the divergence. The resulting terms cancel out due to Clairaut's Theorem.

2. Line integrals

(1) **Question: How does one know when to apply the Fundamental Theorem of Line Integrals (FTLI)?**
Answer: If you are being asked to compute a line integral of a complicated vector field, if curve you need to parametrize would be hard to parametrize, or if computing the line integral directly would be difficult, check whether the vector field is conservative. If and onlify if it is conservative, applying FTLI may make your life easier because you just need to find a potential function and evaluate that function at the endpoints of the curve.

(2) **Question: When is it best to apply Green's theorem?**
Answer: If you are being asked to compute the line integral of a vector field $\vec{F} = (P, Q)$ along a **closed** curve C, it may be easier to apply Green's theorem if the curve C bounds a region which is easy to parametrize as a surface. For example, if you are being asked to do a line integral over the boundary of a rectangle, it may be easier to apply Green's theorem. If we were to compute the line integral directly, we would have to break the integral into four different parts, one for each line segment in the boundary.

(3) **Question: How do you visually determine the sign of the work of a vector field along a curve?**
Answer: The trick is to remember that the work integral of a vector field F along a curve C parametrized by $r(t)$ can be written as:

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt = \int_a^b |F(r(t))|\,|r'(t)|\,\cos(\theta(t))\, dt$$

where $\theta(t)$ is the angle between the vector field $F(r(t))$ and the velocity vector $r'(t)$. Note that the only quantity that changes sign in the expression inside the integral is the term $\cos(\theta(t))$. Specifically, if $0 \leq \theta(t) \leq \pi/2$, that is if the angle between the vector field and the velocity is acute, then we have $\cos(\theta(t)) \geq 0$. On the other hand, if $\pi/2 \leq \theta(t) \leq \pi$, that is if the angle between the vector field and the velocity is obtuse, we have $\cos(\theta(t)) \leq 0$. Thus, the idea to determine the sign of the work visually is to determine whether the vector field is generally acute or obtuse with the velocity. If it is both acute and obtuse along a curve, the magnitudes will affect the net result, but if it is overwhelmingly acute then it's likely positive work, and vis a vis for the obtuse case. On the other hand, note that if $\theta(t) = \pi/2$, that is if the vector field is perpendicular to the velocity, then $\cos(\theta(t)) = 0$. Hence, there is no work wherever the vector field is perpendicular to the motion.

3. Surface integrals

(1) **Question: How does one determine the direction of the normal vector given a parametrization?**
Answer: The normal vectors of a parametrization $r : D \to \mathbb{R}^3$ are given by the cross product $r_u \times r_v$. In general, $r_u \times r_v$ depends on the parameters u and v. One way to determine which way the normal vectors are pointing, it suffices to pick a value of the parameters, say (u_0, v_0), and plot the vector $(r_u \times r_v)(u_0, v_0)$ anchored at the point $r(u_0, v_0)$. For example, in the parameterization $r : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$ of the unit sphere:

$$r(\phi, \theta) = \Big(\cos(\theta)\sin(\phi),\ \sin(\theta)\sin(\phi),\ \cos(\phi)\Big)$$

the normals are:

$$r_\phi \times r_\theta = \Big(\cos(\theta)\sin^2(\phi),\ \sin(\theta)\sin^2(\phi),\ \sin(\phi)\cos(\phi)\Big).$$

Picking the parameters $\phi = \pi/2$ and $\theta = 0$ we get the normal vector $(r_\phi \times r_\theta)(\pi/2, 0) = (1, 0, 0)$ which points outward when anchored at the point $r(\pi/2, 0) = (1, 0, 0)$. Hence, this parametrization induces the outward pointing orientation on the unit sphere. Sometimes we allow parametrizations to be a little bit degenerate at the corners of the parameter domain. It's best to avoid using these to check the normal vectors. In the sphere example, the cross product $r_\phi \times r_\theta$ at the north pole, i.e. $\phi = 0$ and $\theta = 0$, is the zero vector, which doesn't let you see the orientation induced on the rest of the sphere.

(2) **Question: When does one use the magnitude $|r_u \times r_v|$ versus the vector $r_u \times r_v$?**
Answer: In this course we integrate scalar fields and vector fields over surfaces (see question 2 of section 1 above for the difference between scalar and vector fields), and many of the theorems we deal with involve such these integrals. These integrals are defined using parametrizations $r : D \to \mathbb{R}^3$

of the surface. When integrating scalar fields $f : \mathbb{R}^3 \to \mathbb{R}$ on a surface parametrized by r we use the formula:

$$\iint_S f dA = \iint_D f(r(u,v))\,|r_u \times r_v|\,dudv,$$

which involves the magnitude $|r_u \times r_v|$. When integrating the flux of a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ through a surface parametrized by r we use the formula:

$$\iint_S F \cdot dA = \iint_D F(r(u,v)) \cdot (r_u \times r_v)\,dudv,$$

which involves the vector $r_u \times r_v$.

(3) **Question:What are the various ways to notate the surface integral of a vector field?**
Answer: If $\mathbf{F}(x, y, z)$ is a vector field and S is an oriented surface, then the following are equivalent ways to write the flux of $\mathbf{F}$ through S.

$$\begin{aligned}\iint_S \mathbf{F}(x,y,z) \cdot \mathrm{d}\mathbf{A} &= \iint_S \mathbf{F}(x,y,z) \cdot \mathrm{d}\mathbf{S} \\ &= \iint_S \mathbf{F}(x,y,z) \cdot \mathbf{n}\,\mathrm{d}A \\ &= \iint_S \mathbf{F}(x,y,z) \cdot \mathbf{n}\,\mathrm{d}S.\end{aligned}$$

In these equations $\mathbf{n}$ is the unit normal vector to S that points in the correct direction in terms of the orientation of S. If we have a parametrization $\mathbf{r}(u, v)$ for S with domain D, then we have

$$\begin{aligned}\mathbf{n} &= \frac{1}{|\mathbf{r}_u \times \mathbf{r}_v|}\mathbf{r}_u \times \mathbf{r}_v \\ \mathrm{d}A = \mathrm{d}S &= |\mathbf{r}_u \times \mathbf{r}_v|\,\mathrm{d}u\,\mathrm{d}v\end{aligned}$$

and so

$$\mathrm{d}\mathbf{A} = \mathrm{d}\mathbf{S} = \mathbf{n}\,\mathrm{d}A = \mathbf{n}\,\mathrm{d}S = (\mathbf{r}_u \times \mathbf{r}_v)\,\mathrm{d}u\,\mathrm{d}v.$$

(4) **Question: How can you tell whether the surface integral of a vector field is positive or negative?**
Answer: This is similar to Question 2-(3). Remember that surface integrals measure the magnitude of a vector field in the normal direction of a surface. At each point measure the angle between the normal vector of your surface and vector field. If you have an acute angle on a majority of the points it is likely that your surface integral will be positive. If you have an obtuse angle on a majority of the points it is likely that your surface integral will be negative. Roughly speaking, if your vector field lies in the same direction as the normal vector then your surface integral is positive and vice versa.

4. Green's, Stokes', Gauss', and Divergence Theorems

(1) **Question: When should one use Stokes' theorem?**
Answer: If we are asked to compute the line integral over a **closed** curve C of a vector field $\vec{F}$, we may want to consider applying Stokes' theorem if the flux integral of curl(F) over a surface with boundary curve C is simpler. This may happen because curl(F) is a simpler vector field or because its restriction to a surface with boundary C is simple.

(2) **Question: How do you determine the orientation of the boundary curve when using Stokes' theorem given the normal vector?**
Answer: Imagine yourself standing near the boundary of the surface with your head in the direction of the normal, then the orientation of the curve is given by the direction of you r walking such that the surface is to your left.

(3) **Question: How do you determine the orientation of the surface when using Stokes' theorem given the orientation of the boundary curve?**
Answer: Imagine yourself standing near the boundary of the surface following the direction of the boundary curve. You should walk such that the surface is to your left, the direction your head is pointing such that this happens is the direction of the normal vector.

(4) **Question: How does Gauss' Law work? What is its relationship to the divergence theorem?**
Answer: Gauss' Law is a special application of the Divergence Theorem. Gauss' Law tells us the flux of a vector field

$$\begin{aligned}\mathbf{F}(x,y,z) &= \frac{1}{(x^2+y^2+z^2)^{3/2}}\langle x,y,z\rangle \\ &= \frac{1}{|\mathbf{r}|^3}\mathbf{r}\end{aligned}$$

through the boundary ∂E of some region E containing the origin is

$$\iint_{\partial E} \mathbf{F}(x,y,z)\cdot \mathrm{d}\mathbf{S} = 4\pi.$$

Note that $\operatorname{div}\mathbf{F}(x,y,z) = 0$ This is valid for $(x,y,z) \neq (0,0,0)$ since the vector field is not defined at the origin. Thus, the Divergence Theorem tells us that the flux of $\mathbf{F}(x,y,z)$ through the boundary ∂E of a region E that does not contain the origin is zero. We can put these two together to get the following. If we have a point $\mathbf{r}_0$ and a region E, then the flux of the vector field:

$$\mathbf{F}(\mathbf{r}) = \frac{Q}{|\mathbf{r}-\mathbf{r}_0|^3}(\mathbf{r}-\mathbf{r}_0)$$

through the boundary of E is:
- $4\pi Q$ if $\mathbf{r}_0$ is inside E, and
- zero otherwise.

(5) **Question: When should one use the divergence theorem?**
Answer: Suppose you are asked to take the surface integral of a vector field $\vec{F}$ over some surface S which bounds a **closed** region E in $\mathbb{R}^3$. In this case, it may be easier to apply the divergence theorem and evaluate $\iiint_E \operatorname{div}\vec{F}\, dV$. This is especially true when the surface S consists of multiple components (e.g. a hemisphere and a disk, or 6 faces of a box...) or when the divergence of $\vec{F}$ is especially simple (a constant or function which is easy to integrate).