MIDTERM 1 FAQ

(1) If a question asks for the vector perpendicular to the given plane, are there multiple possible answers (vectors) to this problem?
Answer: Yes. They can differ by magnitude.

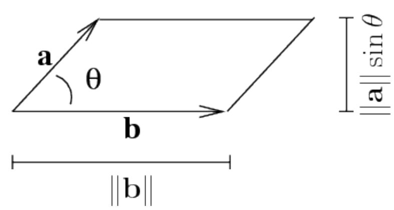
(2) What are the standard axes labels on exams? Some graphs are in a box, and the axes are not labeled. What should we assume the axes are? How can we answer these questions?
Answer: If there is truly no information telling you what the axes should be, that probably means that information is not strictly necessary for the question. It is probably safe to assume that if the graph is not labelled at all, then the z-axis points in the vertical direction. However, you should not assume which axis is the x-axis and which is the y-axis.

(3) What is the best way to go about matching equations to graphs?
Answer: On many of the "matching a function with its graph" problems, a good way to start is process of elimination. For example, the function $\sin(x)\sin(y)$ is zero along the x-axis and the y-axis. (This is because $\sin(0) = 0$.) This should narrow it down: we can rule out any graph which is not zero along these axes. If you're left with only two graphs, compare each graph and find a property that one has and the other doesn't; then check that property against the function. For example, if one graph has $f(1,2) < 0$ and the other has $f(1,2) > 0$, you should be able to rule one out by checking if your function is positive or negative at the point $(1,2)$.

(4) Why is the magnitude of the cross product of $\mathbf{a}$ and $\mathbf{b}$ equal to the area of the parallelogram?
Answer: Consider the identity:

$$||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||\,||\mathbf{b}||\sin(\theta)$$

and the following image:



Then apply the usual formula for the area of a parallelogram.

(5) How do I find the sign of $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$ using contour plots?
Answer: Note that $\frac{\partial f}{\partial x}$ is the rate of change of f in the direction of x. You can use this fact and the contour plot to determine the sign. For example, say we are asked to find the sign of $\frac{\partial f}{\partial x}$ at a point A. Imagine yourself walking through the point A parallel to the x-axis and in the positive direction. Compare the values of the level sets of f immediately before and after walking through the p oint A. If the values of the level sets increase, then $\frac{\partial f}{\partial x}(A) > 0$. If the values of the level sets decrease, then $\frac{\partial f}{\partial x}(A) < 0$. If the values of the level sets return to the same value immediately after A that were obtained immediately before A, then $\frac{\partial f}{\partial x}(A) = 0$. The process is similar for $\frac{\partial f}{\partial y}$ except using the y-axis instead of the x-axis.

(6) How do I find the sign of $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$ at a point using the graph of the function f?
Answer: If, for example, we are trying to find $\frac{\partial f}{\partial y}$ at a point A, we can take a "cut" in the direction of (parallel to) the y-axis through the point A. In the cross section of the cut, we would see a curve that goes through A. Look at the slope of the tangent line to that curve at the point A. If the slope

is positive, then $\frac{\partial f}{\partial y}(A)$ is positive. If the slope is negative, then $\frac{\partial f}{\partial y}(A)$ is negative. If the slope is zero, then $\frac{\partial f}{\partial y}(A)$ is zero. See 2011 midterm 1 #5.

(7) How do I find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, and $\frac{\partial^2 f}{\partial x \partial y}$ using contour plots?
Answer: Rewriting, we get:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial x}\Big), \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial y}\Big)$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}\Big) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial x}\Big).$$

So, for example, given a point P, we can interpret $\frac{\partial^2 f}{\partial x \partial y}(P)$ as the rate of change of $\frac{\partial f}{\partial y}(P)$ as we move through the P parallel to the x-axis in the positive direction. We can use this and the contour plots to determine the sign of the second derivative. Basically check the slope of the tangent line to the cross-section for y fixed right before P (before in the x-direction), at P, and right after P. If the slopes are increasing, then $\frac{\partial^2 f}{\partial x \partial y}(P) > 0$. If the slopes are decreasing, then $\frac{\partial^2 f}{\partial x \partial y}(P) < 0$. If the slopes are return to the same value right after P that they had before P, then $\frac{\partial^2 f}{\partial x \partial y}(P) = 0$.

For the derivatives $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y y^2}$, you can also use the spacing between the level sets as follows. Given a point P, you can determine the sign of $\frac{\partial^2 f}{\partial x^2}(P)$ by traveling and seeing how the distance between the level sets, measured in the x direction, and the values of the function change. If the gaps between the level sets decrease as we move through P and the values of the level sets increase, then $\frac{\partial^2 f}{\partial x^2}(P) > 0$. If the gaps between the level sets decrease as we move through P and the values of f increase, then $\frac{\partial^2 f}{\partial x^2}(P) < 0$. If the gaps between the level sets increase as we move through P, and the values of f increase, then $\frac{\partial^2 f}{\partial x^2}(P) < 0$. If the gaps between the level sets increase as we move through P, with the values of f decrease, then $\frac{\partial^2 f}{\partial x^2}(P) > 0$. The other case is similar.

(8) How does the right hand rule for cross products work?
Answer: To find the direction of the resultant vector of $\mathbf{a} \times \mathbf{b}$, place your right hand on the first vector of the cross product, in our case $\mathbf{a}$, with your hand upright. Then curl your fingers (in the shortest direction) towards the second vector in the cross product, in this case $\mathbf{b}$. After that, your thumb will point in the direction of the resultant vector, $\mathbf{a} \times \mathbf{b}$.

(9) What are some tips to visualize graphs of functions of two variables? How can we identify graphs on multiple choice questions?
Answer: Look at each function and try to find x and y values for where the function will be a max, a min, or zero. For example $\frac{1}{1+x^2+y^2}$ is gonna have a max at $x = 0, y = 0$. Additionally, you can look for lines that would make the function 0 such as the case with $\sin(x - y)$ for $y = x$(more generally $y = x + k\pi$ for integer k). Also check to see if the function is periodic with respect to x or y or both. You can also look at the intersection of the graph with the planes $x = 0$ and $y = 0$, get the contour map by looking at intersections with the planes $z = k$ and even look at the behaviour as you approach infinity. It also helps to try to solve these kinds of problems via elimination, identifying key features via the above tips to rule out options.

(10) Is the equation for a tangent plane to a graph of a function at a point p essentially the same as the linear approximation of the function at that point?
Answer: The tangent plane to a graph of a function at a point p is the graph of the linear approximation at p (also called the linearization). Thus, the equation for the tangent plane is $z = L(x, y)$, where $L(x, y)$ is the linearization at the point p.

(11) Can two lines parallel to the same plane NOT be parallel?
Answer: Imagine a triangle on a 2d plane (for example, on the plane $\mathbb{R}^2$). All three sides of the triangle are parallel to the plane, but none of them are parallel to each other.

(12) Question 1. (e) of practice Midterm 1 from 2010 asks to determine whether a given line is parallel to the plane containing three given points. In the answer key it says that the line is parallel to the plane because the dot product of a velocity vector for the line and a normal vector to the plane equals zero. Shouldn't this mean that the line is perpendicular to the plane? Because everywhere in the book it says that two vectors are orthogonal if the dot product equals 0.

Answer: The line is perpendicular to the *normal* vector of the plane, not to the plane itself. Any line perpendicular to the normal vector of a plane will be parallel to that plane, since (after translation) the plane essentially contains *all* vectors perpendicular to the normal vector.

(13) How do you know when to use polar coordinates to prove a limit as $(x, y) \to (0, 0)$ exists or doesn't exist. Can you prove a limit does not exist by showing the limit along two different paths is not the same? Can you prove it exists by showing the limit along two lines is equal?

Answer: Using polar coordinates (r, θ) can simplify checking a limit, since if the limit exists then it should exist as $r \to 0$ regardless of the value of θ. If you calculate the limit as $r \to 0$ and get an answer that depends on θ, then you can show the limit does not exist by choosing values of θ to make the answer different. For example, you could choose, $\theta = 0$ and $\theta = \pi/4$, which correspond to the x-axis and the line $y = x$.

To check if a limit does not exist it also suffices to just check along two paths and if the results is not the same, then the limit does not exist. However, you cannot use this to prove a limit exists.

(14) Question 4 (c) from 2012 Midterm 1 gives a plane A with equation $x - z = 1$, a plane B given by the equation $x + y + z = 2$. It asks us to find the equation of a plane C which is perpendicular to both A and B. I can find the normal vector to C by taking the cross product of normal vectors for A and B. However, why do they use the point $(0, 0, 0)$ in the answer key. How do we know the point $(0, 0, 0)$ is in the plane C? Also, can there be such a plane C that is perpendicular to both A and B?

Answer: You can choose any point to lie on C, choosing a different point will only translate C along the line spanned by the normal vector, so it will still be orthogonal to A and B. Finding a plane orthogonal to two given ones is geometrically possible. To see why, note that two planes in $\mathbb{R}^3$ are orthogonal if they have orthogonal normal vectors. Hence the problem is equivalent to finding a third vector orthogonal to two given vectors in $\mathbb{R}^3$, which is certainly possible. To further convince yourself, think note that the planes $x = 0$, $y = 0$, and $z = 0$ are all mutually orthogonal to each other.

(15) I don't understand the last line of the solution to question 13 of the 2012 midterm 1. Why does this prove f can't be differentiable at $(0, 0)$?

Answer: In order for f to be differentiable at $(0, 0)$, there needs to be a tangent plane at this point. This means that we need $\lim_{(x,y)\to(0,0)} \frac{E(x,y)}{\sqrt{x^2+y^2}} = 0$, where $E(x, y)$ is our error function, or how far our approximation is away from the value of the function. As the solution shows, considering the limit along the line $x = y$, which we can parametrize as (t, t) for t a real number, yields a different limit. Thus, the limit we need to be 0 is not 0, meaning the function is not differentiable.