1. Consider the vectors: $\mathbf{a} = \langle 1, 0, -1 \rangle$ $\quad \mathbf{b} = \langle 1, 1, 1 \rangle$ $\quad \mathbf{c} = \langle -1, 1, 0 \rangle$.

(a) Compute $\mathbf{a} \times \mathbf{b}$. **(3 points)**

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \vec{k}$$

$$= \vec{i} - 2\vec{j} + \vec{k}$$

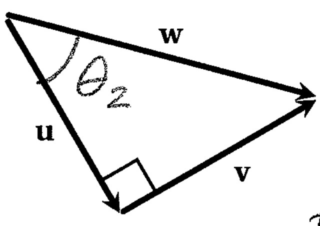
$\boxed{\mathbf{a} \times \mathbf{b} = \langle 1, -2, 1 \rangle}$

(b) Compute the volume of the parallelepiped determined by the vectors **a**, **b**, and **c**. **(2 points)**

$$\text{Vol} = |\vec{c} \cdot (\vec{a} \times \vec{b})| = |\langle -1, 1, 0 \rangle \cdot \langle 1, -2, 1 \rangle|$$
$$= |-1 - 2 + 0| = |-3| = 3$$

$\boxed{\text{Volume} = 3}$

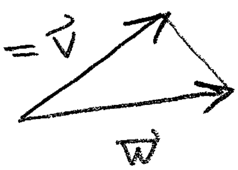
2. Given that **u** and **v** in the picture at left have length 1, compute $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \cdot \mathbf{w}$, and $\text{proj}_{\mathbf{v}}\mathbf{w}$. **(1 point each)**



$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta_1 = 1 \cdot 1 \cdot 0$$
$$\vec{u} \cdot \vec{w} = |\vec{u}||\vec{w}| \cos\theta_2 = 1 \cdot \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1$$

$\text{proj}_{\vec{v}} \vec{w} =$ component of $\vec{w}$ along $\vec{v}$
$= \vec{v}$

$\text{proj}_{\vec{v}} \vec{w} = \vec{v}$



$\boxed{\mathbf{u} \cdot \mathbf{v} = 0}$ $\quad \boxed{\mathbf{u} \cdot \mathbf{w} = 1}$ $\quad \boxed{\text{proj}_{\mathbf{v}}\mathbf{w} = \vec{v}}$

3. A particle moves with constant velocity $\langle 3, 1, -1 \rangle$ starting from the point $(3, 2, 4)$ at time $t = 0$. When and where will it cross the xy-plane? **(3 points)**

$$\vec{r}(t) = (3, 2, 4) + t\langle 3, 1, -1 \rangle = (3 + 3t, 2 + t, 4 - t)$$

So particle hits xy-plane when z-coor is 0, that is $4 - t = 0 \Rightarrow t = 4$. At that time, the position is $\vec{r}(4) = (3 + 3 \cdot 4, 2 + 4, 0)$
$= (15, 6, 0)$

$\boxed{\text{When: } t = 4}$ $\quad \boxed{\text{Where: } (15, 6, 0)}$

4. Let A be the plane given by $x - z = 1$ and B the plane given by $x + y + z = 2$.

(a) Find a normal vector $\mathbf{n}$ for the plane A. **(1 points)**

Read off from eqn.

$\boxed{\mathbf{n} = \langle 1, 0, -1 \rangle}$

(b) Find the angle between the two planes. **(2 points)**

$\vec{m}$ = normal to $B = \langle 1, 1, 1 \rangle$

$\vec{n} \circ \vec{m} = 0 \implies \vec{n} \perp \vec{m}$.

$\boxed{\theta = \pi/2}$

(c) Find the equation of a plane C which is perpendicular to both A and B. **(3 points)**

Want normal $\vec{c}$ for C to be $\perp$ to $\vec{n}$ and $\vec{m}$.

So can take $\vec{c} = \vec{n} \times \vec{m} = \langle 1, -2, 1 \rangle$

Pick the C which passes through $(0,0,0)$,

which is given by $1 \cdot (x - 0) - 2(y - 0) + 1(z - 0) = 0$,

that is

Equation: $\boxed{1}\,x + \boxed{-2}\,y + \boxed{1}\,z = \boxed{0}$

5. Exactly one of the following two limits exists. Circle the one that exists and justify your answer. **(5 points)**

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^2 - y^2}{\sqrt{x^2 + y^2}} \right) \qquad \lim_{(x,y)\to(0,0)} \left(\frac{xy}{\left(x^2 + y^2\right)^2} \right)$$

In polar coordinates, $f = \dfrac{x^2 - y^2}{\sqrt{x^2 + y^2}} = \dfrac{r^2\cos^2\theta - r^2\sin\theta}{r}$

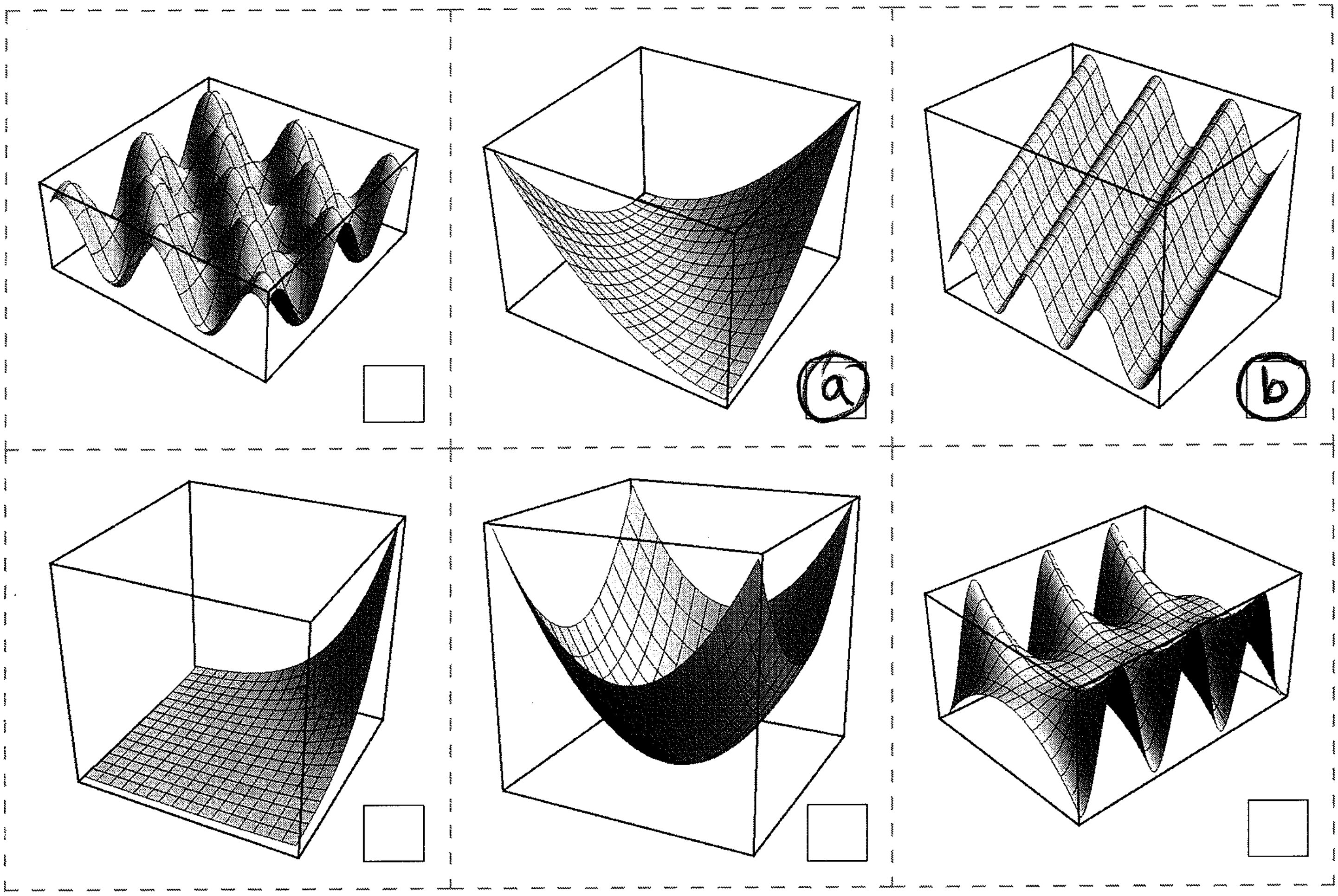
$= r(\cos^2\theta - \sin^2\theta)$. So as $|\cos^2\theta|, |\sin^2\theta| < 1$,

$0 \leq |f| = |r(\cos^2\theta - \sin^2\theta)| \leq r$

Since $r \to 0$ as $(x,y) \to 0$, we must

have $|f| \to 0$ and hence $\lim\limits_{(x,y)\to(0,0)} f = 0$.

6. For each function label its graph from among the options below: (a) $(x+y)^2$ (b) $x+\cos(y)$
(3 points each)



7. Circle the equation for the quadratic surface shown at right. **(3 points)**

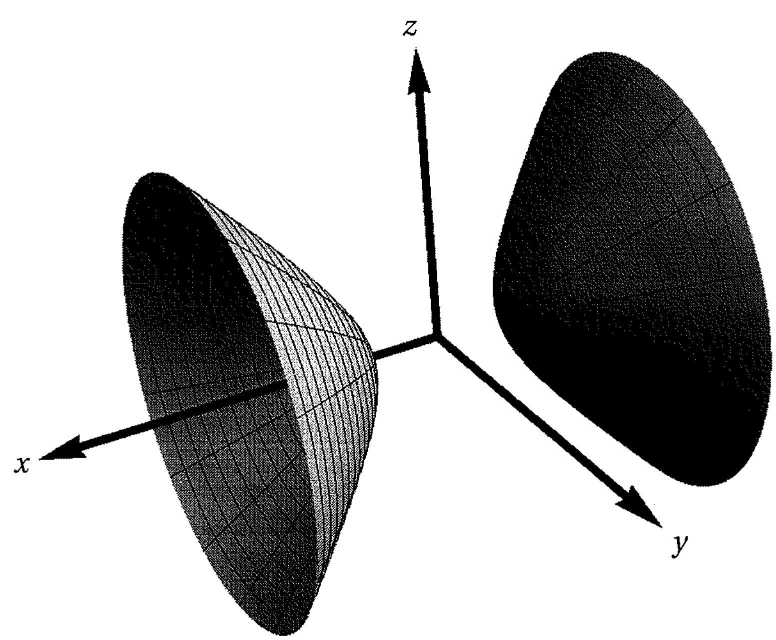
(a) $x^2+y^2+z^2=1$

(b) $x^2-y^2-z^2=-1$

(c) $x^2+y^2-z^2=-1$

(d) $x^2-y^2-z^2=1$

(e) $x-y^2-z^2=1$



(6) (a) $f(x,y) = (x+y)^2$

Take vertical cross sections and use process of elimination

In particular, note that:

if $x=0$: $f(0,y) = y^2$ $\leftarrow$ parabola

if $y=0$: $f(x,0) = x^2$ $\leftarrow$ parabola

if $y=x$: $f(x,x) = 4x^2$ $\leftarrow$ parabola

if $y=-x$: $f(-x,x) = (-x+x)^2 = 0$ $\leftarrow$ horizontal line

More generally, if k is a constant, then:

$$f(kx,x) = (kx+x)^2 = (k+1)^2 x^2,$$

which has graph a parabola (except for $k=-1$).

Thus, we expect vertical cross sections for the vertical planes $y = kx$ through the origin to be parabolas that open up at different rates (except for $k=-1$, i.e. $y=-x$, which is a horizontal line). The only choice that satisfies this is the top-middle graph.

(b) $f(x,y) = x + \cos(y)$

Note that along the vertical planes $x = k$, where k is a constant, the function becomes $f(k,y) = k + \cos(y)$. Thus, these vertical cross sections have graphs that are the same as that of cosine but shifted up or down. By process of elimination, only the top right satisfies this.

(7) Note that the surface has a gap at $x=0$. Hence, we seek an equation that has no solutions for $x=0$. Substituting $x=0$ into the choices yields:

(a) $y^2+z^2=1$
(b) $-y^2-z^2=-1$
(c) $y^2-z^2=-1$
(d) $-y^2-z^2=1$
(e) $-y^2-z^2=1$

Choices (a), (b), and (c) have solutions so we are left with (d) and (e). However, note that (e) has no solutions for $x<0$ since then $x-y^2-z^2$ would be negative but the right hand side 1 is positive. The surface has points with $x<0$, so the equation must be (d).

8. Is the function $f: \mathbb{R}^2 \to \mathbb{R}$ given at right continuous at $(0,0)$? Justify your answer. **(2 points)**

$$f(x,y) = \begin{cases} x^2+y+1 & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

No since

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} x^2+y+1 = 1 \neq f(0,0) = 0.$$

since x^2+y+1 is cont

9. Let $f(x,y)$ be a function with values and derivatives in the table. Use linear approximation to estimate $f(2.1, 0.9)$. **(3 points)**

(x,y)	$f(x,y)$	$\frac{\partial f}{\partial x}(x,y)$	$\frac{\partial f}{\partial y}(x,y)$
$(-1,3)$	0	4	4
$(2,1)$	2	-1	3
$(2,4)$	3	7	7
$(3,6)$	1	-3	-5

Do linear approx at $(2,1)$ with $\Delta x = 0.1$ and $\Delta y = -0.1$. Thus

$$f(2.1, 0.9) \approx f(2,1) + f_x(2,1)\Delta x + f_y(2,1)\Delta y$$
$$= 2 + (-1)(0.1) + 3(-0.1)$$
$$= 1.6$$

$f(2.1, 0.9) \approx$ $\boxed{1.6}$

10. Suppose $f(x,y)$ has the contour plot below right, with points labelled. Circle the best answer to each of the following questions: **(1 point each)**

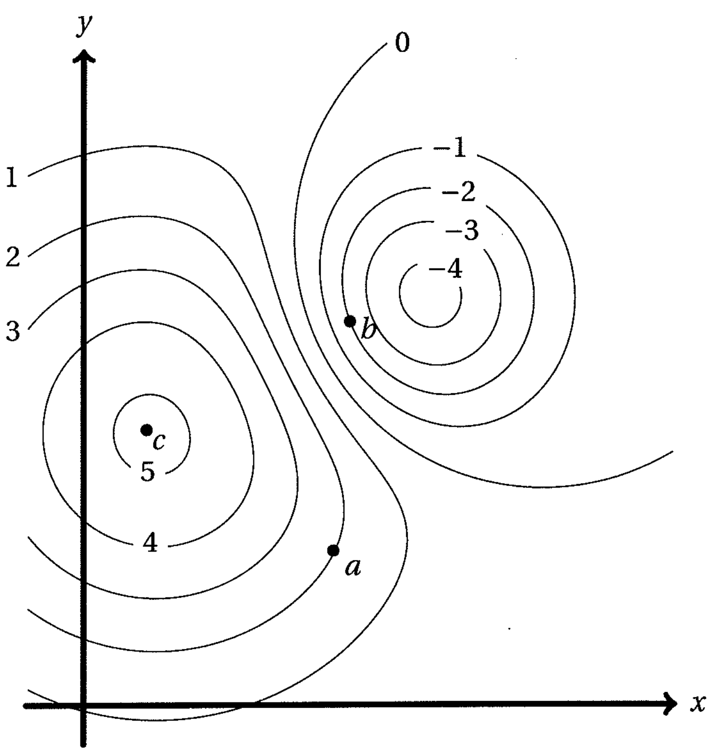
(a) $f(a)$ is: positive negative 0

(b) $\frac{\partial f}{\partial x}(b)$ is: positive negative 0

(c) $\frac{\partial^2 f}{\partial^2 y}(c)$ is: positive negative 0

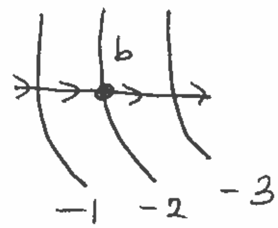
(d) Circle one:

$\frac{\partial f}{\partial x}(a) > \frac{\partial f}{\partial x}(b)$ $\quad$ $\frac{\partial f}{\partial x}(a) < \frac{\partial f}{\partial x}(b)$



(10) (a) The point a is on the level curve $f(x,y)=2$, so $f(a)>0$.

(b) Move parallel to the x-axis in the positive direction at the point b:

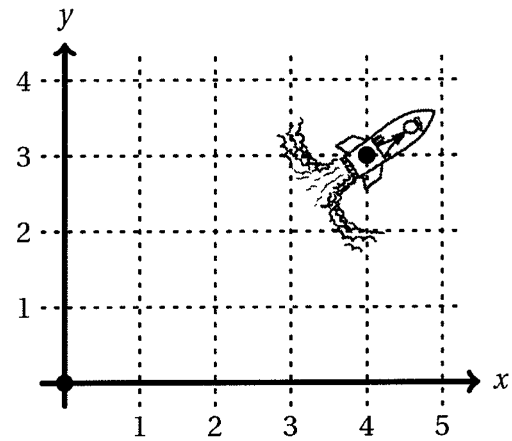


Since the values are decreasing in this direction, then $\frac{\partial f}{\partial x}(b)<0$.

(c) By looking at the nearby level curves, we can see there is a maximum at the point c. Hence, the concavity of the vertical cross section for x fixed passing through the point c is negative at the point c. That is, $\frac{\partial^2 f}{\partial y^2}(c)<0$.

(d) Proceed as in part (b) to note that the function is decreasing in the positive x-direction at both a and b. Hence, both partial derivatives are negative. However, the decrease is faster at b since the level curves are packed more tightly at b. This means the rate of decrease $\frac{\partial f}{\partial x}(b)$ of f at b is more negative than the rate of decrease $\frac{\partial f}{\partial x}(a)$ of f at a. Hence, $\frac{\partial f}{\partial x}(a)>\frac{\partial f}{\partial x}(b)$.

11. An exceptionally tiny spaceship positioned as shown is travelling so that its x-coordinate increases at a rate of $1/2$ m/s and y-coordinate increases at a rate of $1/3$ m/s. Use the Chain Rule to calculate the rate at which the distance between the spaceship and the point $(0,0)$ is increasing. **(6 points)**



Distances in meters

Rocket courtesy of xkcd.com

$D = \text{dist} = \sqrt{x^2+y^2}$

With ship at $(4,3)$, $D=5$.

Now $\frac{\partial D}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \frac{x}{D}$

and $\frac{\partial D}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2}(2y) = \frac{y}{D}$.

By the chain rule:

$$\frac{dD}{dt} = \frac{\partial D}{\partial x}\frac{dx}{dt} + \frac{\partial D}{\partial y}\cdot\frac{dy}{dt} = \left(\frac{4}{5}\right)\cdot\left(\frac{1}{2}\right) + \left(\frac{3}{5}\right)\cdot\left(\frac{1}{3}\right)$$

$$= \frac{2}{5} + \frac{1}{5} = 3/5$$

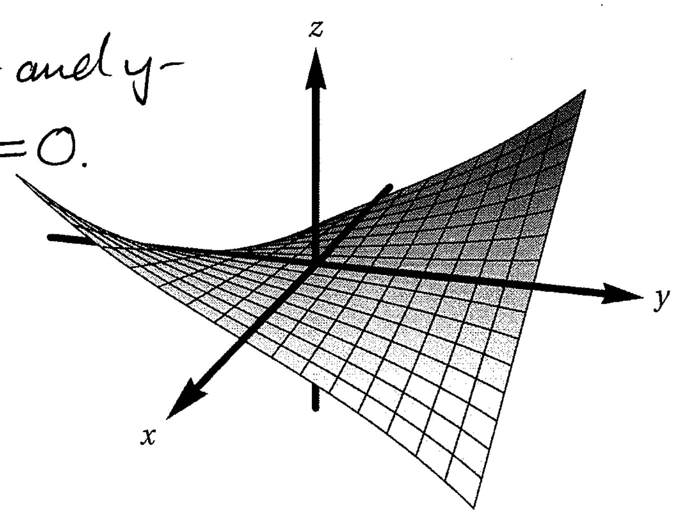
rate = $3/5$ m/s

12. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function whose graph is shown at right.

(a) Find the equation of the tangent plane to the graph at $(0,0,0)$. **(2 points)**

Since the graph contains the x- and y-axes, we have $f_x(0,0) = f_y(0,0) = 0$.

Thus the tangent plane is $\boxed{z=0}$



(b) The partial derivative $\frac{\partial^2 f}{\partial x \partial y}(0,0)$ is (circle your answer):

positive **negative** **0** **(1 point)**

13. Extra Credit Problem. Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(0,0)$ with $f(0,0) = 2$ and partial derivatives $f_x(0,0) = 1$ and $f_y(0,0) = -1$. If in addition

$$\lim_{t\to 0} \frac{f(t,t)-2}{t} = 1$$

can f be differentiable at $(0,0)$? Carefully justify your answer. **(3 points)**

Recall: $f(a+h, b+k) = f(a,b) + \frac{\partial f}{\partial x}(a,b)h + \frac{\partial f}{\partial y}(a,b)k + E(h,k)$.

If f is differentiable at $(0,0)$, we would have

$$\lim_{(h,k)\to(0,0)} \frac{E(h,k)}{\sqrt{h^2+k^2}} = 0.$$

If $(a,b) = (0,0)$, then

$$f(h,k) = 2 + h - k + E(h,k).$$

and if $h = k = t$,

$$f(t,t) - 2 = E(t,t).$$

Then notice

$$\lim_{t\to 0} \frac{E(t,t)}{\sqrt{t^2+t^2}} = \lim_{t\to 0} \frac{E(t,t)}{\sqrt{2}\,t} = \frac{1}{\sqrt{2}} \lim_{t\to 0} \frac{f(t,t)-2}{t} = \left(\frac{1}{\sqrt{2}}\right)(1) = \frac{1}{\sqrt{2}} \neq 0.$$

So the limit

$$\lim_{(h,k)\to(0,0)} \frac{E(h,k)}{\sqrt{h^2+k^2}} \neq 0,$$ and hence $\boxed{\text{f cannot be differentiable at } (0,0)}$

Scratch work may go below and on the back of this sheet.