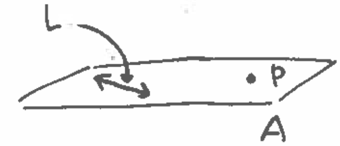
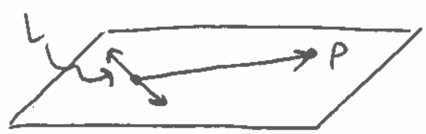
1. Find an equation for the plane containing the point $P = (1,2,3)$ and the line L parametrized by $x = 1 + t$, $y = 2t$, and $z = 2 - t$. **(5 points)**

We want an equation for the plane A shown:



So, we need a point on the plane and a normal vector to the plane. To find a normal vector to the plane, we can take 2 vectors on (or parallel) to the plane. One choice for a vector is the velocity vector $\vec{v} = \langle 1, 2, -1 \rangle$, since it is parallel to the line L. Next, we can choose any vector that starts at a point on L and ends at P, as shown:



So, a point on L is $Q = (1, 0, 2)$ when $t = 0$. Hence, the vector $\vec{w} = \vec{QP} = (1,2,3) - (1,0,2) = \langle 0, 2, 1 \rangle$. So, a vector normal to A is

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} k$$

$$= [(2)(1) - (2)(-1)]i - [(1)(1) - (0)(-1)]j + [(1)(2) - (0)(2)]k$$

$$= 4i - j + 2k.$$

So, the equation of the plane is

$$4(x-1) - (y-2) + 2(z-3) = 0$$

$$\Longleftrightarrow 4x - y + 2z = 8$$

Equation: $\boxed{4}\, x + \boxed{-1}\, y + \boxed{2}\, z = \boxed{8}$

2. Consider the vectors $\mathbf{a} = \langle 0, 1, 1\rangle$ and $\mathbf{b} = \langle -1, 0, 1\rangle$.

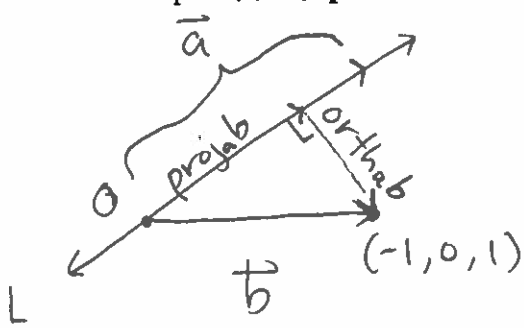
(a) Find $\text{proj}_{\mathbf{a}}\mathbf{b}$, the vector projection of **b** onto **a**. **(2 points)**

$$\text{proj}_a b = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a} = \frac{(0)(-1)+(1)(0)+(1)(1)}{(0)^2+(1)^2+(1)^2}\langle 0,1,1\rangle$$

$$= \frac{1}{2}\langle 0,1,1\rangle = \langle 0, \tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$\boxed{\text{proj}_{\mathbf{a}}\mathbf{b} = \langle 0, \tfrac{1}{2}, \tfrac{1}{2}\rangle}$

(b) Find the distance from the point $(-1,0,1)$ to the line through the origin whose direction vector is **a**. Hint: Use part (a)! **(3 points)**



So the distance is the magnitude of $\text{orth}_a b$.

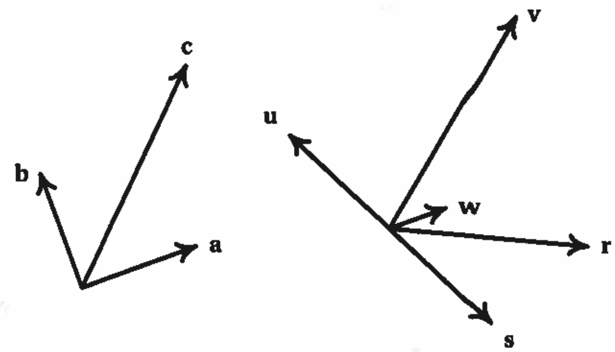
$$\text{orth}_a b = \vec{b} - \text{proj}_a b = \langle -1,0,1\rangle - \langle 0,\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$= \langle -1, -\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$|\text{orth}_a b| = \sqrt{(-1)^2+\left(-\tfrac{1}{2}\right)^2+\left(-\tfrac{1}{2}\right)^2} = \sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}}$$

$\boxed{\text{distance} = \frac{\sqrt{3}}{\sqrt{2}}}$

3. Consider the vectors **a**, **b**, and **c** shown at right. Their lengths are $|\mathbf{a}| = |\mathbf{b}| = 1$ and $|\mathbf{c}| = 2$.
(1 point each)



(a) Circle the vector that best represents $\frac{1}{2}\mathbf{a} - \mathbf{b}$.

u	**v**	**w**	**r**	**s**

(b) Circle the number closest to $\mathbf{a} \cdot \mathbf{c}$.

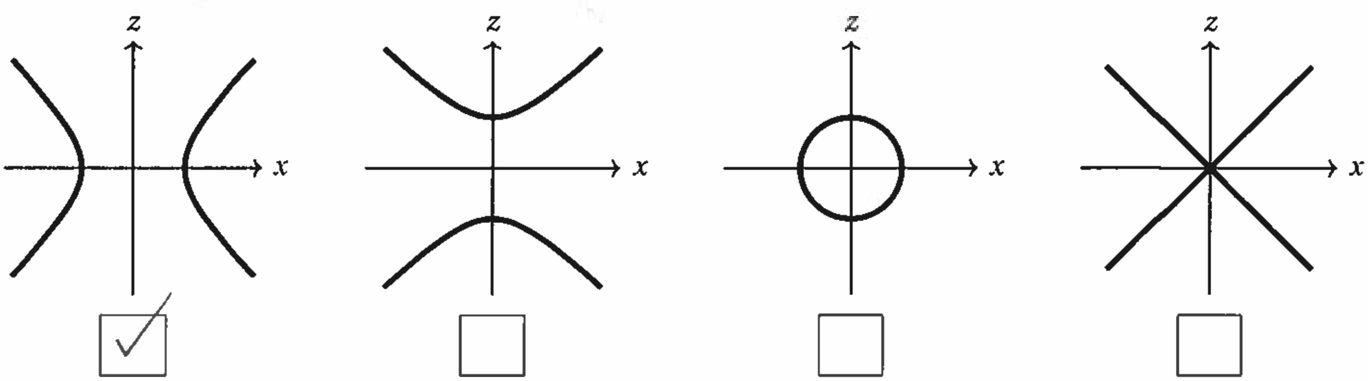
0	1	$\sqrt{2}$	$\sqrt{3}$	2

Explanation on separate page

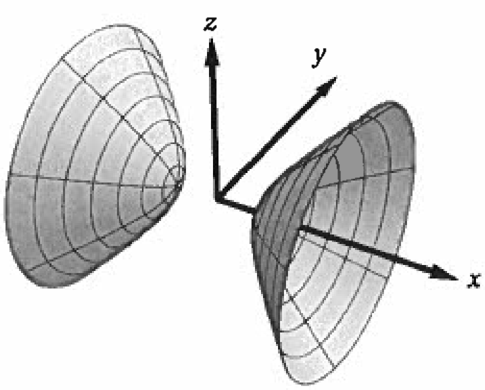
(c) A nonzero vector **n** is pointing directly up out of the page. Circle the best description of $\mathbf{b} \times \mathbf{c}$.

it is a positive multiple of **n**	it is a negative multiple of **n**	it is not a multiple of **n**

4. Consider the quadric surface Q defined by the equation $x^2 + y^2 - z^2 = 1$. Check the box below the picture of the curve formed by intersecting Q with the xz-plane. **(2 points)**



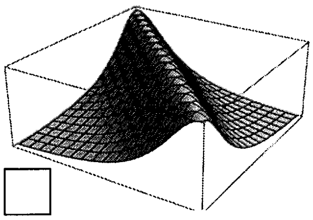
5. Consider the hyperboloid H of two sheets shown below. Circle the equation of H. **(2 points)**

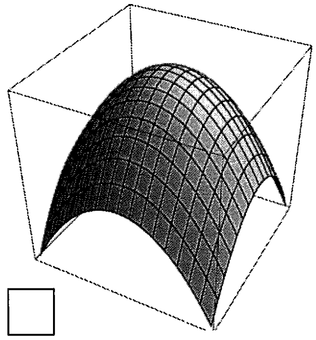


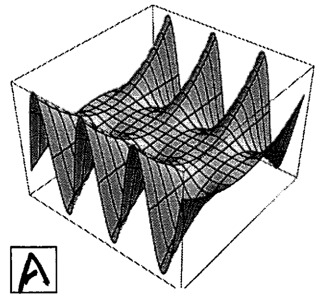
$y^2 + z^2 - x^2 = 1$	$x = y^2 + z^2 + 1$
$x^2 + y^2 + z^2 = -1$	$x^2 - y^2 - z^2 = 1$

6. For each function, label its graph from among the options below by writing the corresponding letter in the box next to the graph. **(2 points each)**

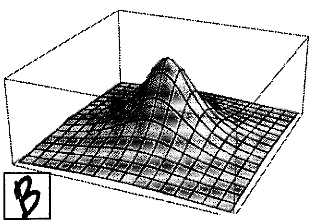
(A) $-y^2 \sin(x)$ (B) $\dfrac{1}{1+x^2+y^2}$



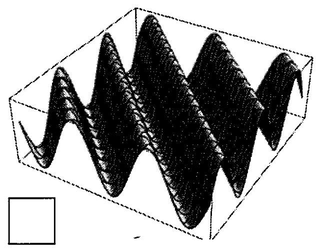


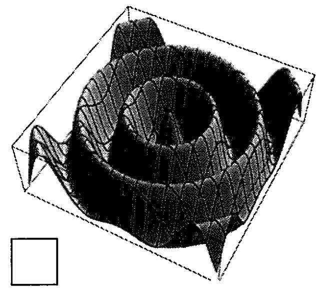


A

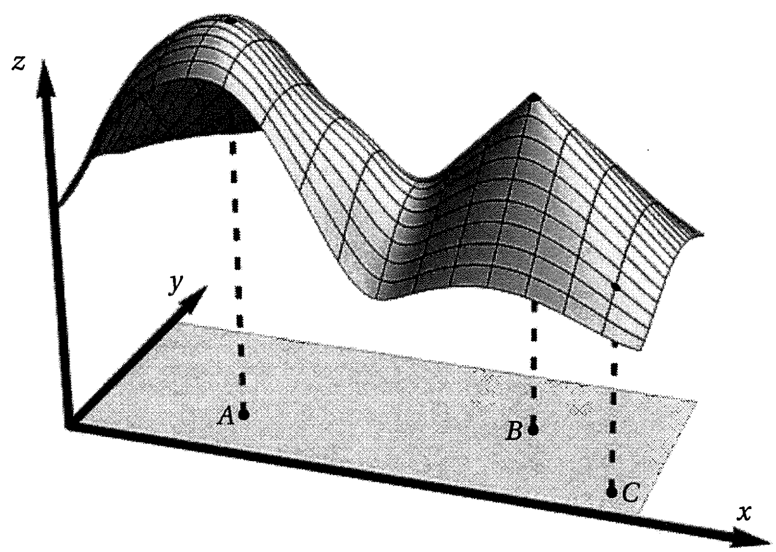


B





7. Consider the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ whose graph is shown at right. Let A, B, and C be the points in $\mathbb{R}^2$ shown; for each of these points, the corresponding point above it on the graph is indicated by the dotted lines and the mark on the graph itself. For each part, circle the answer that is most consistent with the picture. **(1 point each)**



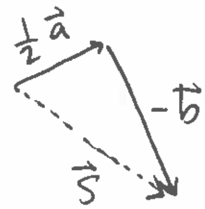
(a) At the point A, the function g is: | continuous | differentiable | both | neither |

(b) At the point B, the function g is: | continuous | differentiable | both | neither |

(c) At the point C, the function $\dfrac{\partial g}{\partial x}$ is: | negative | zero | positive |

3 a. Rewrite $\frac{1}{2}\vec{a} - \vec{b} = \frac{1}{2}\vec{a} + (-\vec{b})$ and use head-to-tail vector addition

$\frac{1}{2}\vec{a}$ =



b. $\vec{a} \cdot \vec{c} = |\vec{a}||\vec{c}|\cos\theta \approx (1)(2)\cos(45°)$

$= \cancel{2} \cdot \frac{\sqrt{2}}{\cancel{2}} = \sqrt{2}$

c. Using the right hand rule to find the direction of $\vec{b} \times \vec{c}$, we get that the resultant vector of $\vec{b} \times \vec{c}$ points into the page, which is opposite to $\vec{n}$. Hence, $\vec{b} \times \vec{c}$ is a negative multiple of $\vec{n}$.

4. For the xz-plane, $y = 0$. So the graph in the xz-plane is

$x^2 - z^2 = 1$

which is a hyperbola in the xz-plane opening in the x-direction.

5. We take cross sections, or cuts, of each possible answer and use process of elimination. Letting $x=0$ (i.e. looking at the cross section in the yz-plane), we see that there should not be a curve. But, this eliminates $y^2+z^2-x^2=1$ since we get the circle $y^2+z^2=1$. Continuing we look at the cross section in the xz-plane. We see that we should get a hyperbola. Letting $y=0$ for the other 3 possibilities, the only equation that gives a hyperbola in the xz-plane is $x^2-y^2-z^2=1$, since when $y=0$,

$$x^2-z^2=1.$$

So, the equation is $x^2-y^2-z^2=1$.

6 a. Again, we take cuts for $z = -y^2 \sin x$.

when $y=0$, $z=0$.

$y=1$, $z=-\sin x$
$y=2$, $z=-4\sin x$ $\Big\}$ as $y \to \pm\infty$, the amplitude of $\sin x$ increases.

By process of elimination, we get the 3rd choice.

b. Taking cuts at $x=0$ & $y=0$, we get $\frac{1}{1+y^2}$ and $\frac{1}{1+x^2}$, respectively. Neither of these functions have an asymptote, and both functions approach 0 as $y \to \infty$ and $x \to \infty$, respectively. This leaves only the 1st and 4th choice. However, taking the cuts at $x=y$ or $x=-y$, we get $\frac{1}{1+2x^2}$, which is non constant. Thus, this eliminates the 1st choice, since on one of $x=y$ or $x=-y$, we get a constant function.

7 a. g is differentiable since the derivative of g is defined (i.e. the slope of every curve on g through the point A exists). In other words, g is "smooth" at A.

b. g is only continuous at B since we see that there is a curve on g through B that has a "corner", like in the function $f(x) = |x|$ at $x = 0$. But, that means the derivative is not defined at B since the slope of the curve at B are two different values as we approach B from both sides.

c. Note that $\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}(g)$, which means we are looking for how g changes as x changes. We can think of cutting g at C in the direction of (parallel to) the x-axis. In the cross section, we see a curve through C. On this curve, we are looking at how the values of g change. If the values increase as we move through C in the positive direction of x, then $\frac{\partial g}{\partial x} > 0$. Similarly, if the values decrease, then $\frac{\partial g}{\partial x} < 0$. If the values don't change, then $\frac{\partial g}{\partial x} = 0$. In this case, the values decrease, so $\frac{\partial g}{\partial x} < 0$.

8. A differentiable function $f: \mathbb{R}^2 \to \mathbb{R}$ takes on the values shown in the table at right.

				x		
		0.2	0.6	1.0	1.4	1.8
	1.8	3.16	3.88	4.60	5.32	6.04
	1.4	2.68	3.24	3.80	4.36	4.92
y	1.0	2.20	2.60	3.00	3.40	3.80
	0.6	1.72	1.96	2.20	2.44	2.68
	0.2	1.24	1.32	1.40	1.48	1.56

(a) Estimate the partials $f_x(1,1)$ and $f_y(1,1)$. **(2 points)**

To get a good estimate, take the nearest points to (1,1) but not at (1,1) since we are approximating points before and after the point. So, $f_x(1,1) \approx \frac{f(1.4,1)-f(0.6,1)}{1.4-0.6} = \frac{3.4-2.6}{0.8} = \frac{0.8}{0.8} = 1$

Note we fix y since we want the change of f only as x changes. Similarly, $f_y(1,1) \approx \frac{f(1,1.4)-f(1,0.6)}{1.4-0.6} = \frac{3.8-2.2}{0.8} = \frac{1.6}{0.8} = 2$

$f_x = \frac{\partial f}{\partial x} \approx \frac{\Delta f}{\Delta x}$, $f_y = \frac{\partial f}{\partial y} \approx \frac{\Delta f}{\Delta y}$

$\boxed{f_x(1,1) \approx 1 \qquad f_y(1,1) \approx 2}$

(b) Use your answer in (a) to approximate $f(1.1, 1.2)$. **(2 points)**

We use linear approximation

$$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$
$$= 3 + (1)(x-1) + 2(y-1) = 3 + (x-1) + 2(y-1)$$
$$L(1.1,1.2) = 3 + (1.1-1) + 2(1.2-1) = 3 + 0.1 + 2(0.2)$$
$$= 3 + 0.1 + 0.4 = 3.5$$

$\boxed{f(1.1,1.2) \approx 3.5}$

(c) Determine the sign of $f_{xy}(1,1)$. **(1 point)**

negative zero positive

Scratch Space

Rewriting, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$. So we see how $\frac{\partial f}{\partial y}$ changes as we move in the x direction. We already know $f_y(1,1)$ so we compare this to $f_y(1.4,1)$

$$\left.\frac{\partial f}{\partial y}\right|_{(1.4,1)} \approx \frac{4.36-2.44}{0.8} = \frac{1.92}{0.8} > 2.$$ So, f_y increases from (1,1) to (1.4,1), hence $f_{xy} > 0$

9. Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ has the table of values and partial derivatives shown at right. For $x(s,t) = s + 2t$ and $y(s,t) = s^2 - t$, let $F(s,t) = f\big(x(s,t), y(s,t)\big)$ be their composition with f. Compute $\dfrac{\partial F}{\partial t}(2,1)$. **(4 points)**

(x,y)	$f(x,y)$	$\dfrac{\partial f}{\partial x}$	$\dfrac{\partial f}{\partial y}$
$(2,1)$	0	7	6
$(2,-1)$	-12	7	-1
$(3,3)$	19	-8	5
$(4,3)$	7	3	2

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

So, finding the above expression for $(s,t) = (2,1)$ we have that $x(2,1) = 4$, $y(2,1) = 3$. Hence,

$$\frac{\partial f}{\partial t}(2,1) = \frac{\partial f}{\partial x}\bigg|_{f(4,3)} \cdot \frac{\partial x}{\partial t}\bigg|_{(2,1)} + \frac{\partial f}{\partial y}\bigg|_{f(4,3)} \cdot \frac{\partial y}{\partial t}\bigg|_{(2,1)}$$

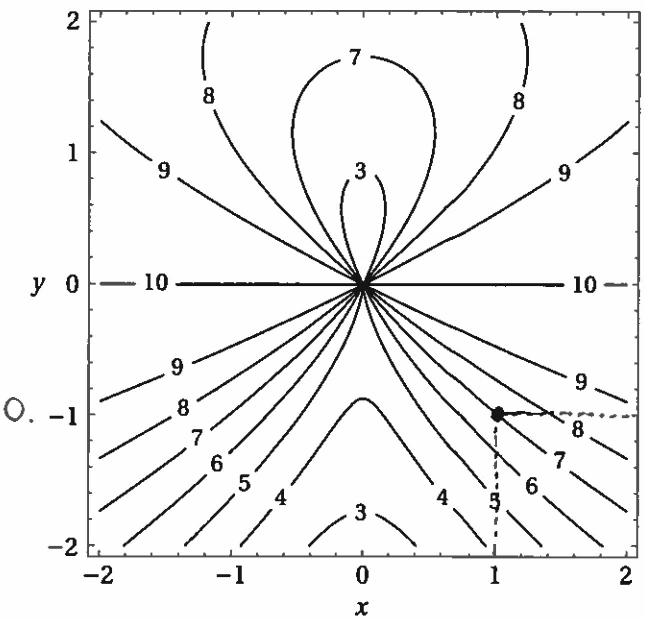
$$= (3)(2) + (2)(-1) = 4$$

since $\dfrac{\partial x}{\partial t}\bigg|_{(2,1)} = 2$ and $\dfrac{\partial y}{\partial t}\bigg|_{(2,1)} = -1$

$\dfrac{\partial F}{\partial t}(2,1) = 4$

Scratch Space

10. Consider the function $f(x, y)$ whose contour map is shown at right, where the value of f on each level curve is indicated by the number along it. For each part, give the answer that is **most consistent** with the given data. **For (a) and (b) be sure to explain your reasoning in the space provided.** If the limit does not exist, write "DNE" in the answer box.



(a) Determine $\lim_{x\to 0} f(x, 0)$. **(2 points)**

As we travel along $y=0$, f is always 10 for all $x \neq 0$. So, the limit is

$\lim_{x\to 0} f(x,0) =$ 10

(b) Determine $\lim_{(x,y)\to(0,0)} f(x, y)$. **(2 points)** If we approach $(0,0)$ on a different path, say the level set $f=9$, we get that the limit is 9. But, since $9 \neq 10$, which is the limit as we travelled on the path $y=0$, then the limit DNE

$\lim_{(x,y)\to(0,0)} f(x, y) =$ DNE

(c) Determine $\lim_{(x,y)\to(1,-1)} f(x, y)$. **(1 point)**

on contour map

$\lim_{(x,y)\to(1,-1)} f(x, y) =$ 7

Scratch Space