

## MIDTERM 2 FAQ

- (1) For min and max questions, when should we use Lagrange multiplier, and when should we use the second derivative test?

**Answer:** It depends on the region:

- (a) If the region is open, one should use the second derivative test.
- (b) If the region is given by some equation constraint  $g(x, y, z) = C$ , one should use Lagrange multipliers.
- (c) If the region is given by  $g(x, y, z) \leq C$  or  $g(x, y, z) \geq C$ , you do the second derivative test for the interior and Lagrange multipliers for the boundary. Then you compare the values.

In this last case if  $g(x, y, z) \leq C$  or  $g(x, y, z) \geq C$  describes a closed and bounded region, then you don't have to do the second derivative test but just find the critical points in the interior (solving  $\nabla f = 0$ ) and the critical points on the boundary (usually using Lagrange multipliers or reducing to a single variable function on the boundary). The Extreme Value Theorem will then guarantee that a max is obtained on one of these points and that a min is obtained on one of these points.

- (2) In discussion section we used Lagrange multipliers to find the maximum of some continuous function restricted to the curve  $x^3 + y^3 = 16$ . We also showed that this curve was not bounded. Doesn't this contradict the Extreme Value Theorem?

**Answer:** No. The Extreme Value Theorem says: if we have a continuous function  $f$  defined on some closed and bounded domain, then  $f$  obtains a maximum and a minimum on this domain. However, it does not say these are necessary conditions as you observed.

- (3) When can we divide by the variables when solving equations in the Lagrange multipliers problems.

**Answer:** Let's look at two examples. First, let  $f(x, y, z) = x^2 + y^2 + z^2$ ,  $g(x, y, z) = x^4 + y^4 + z^4$ . Let the constraint be  $g = 8$ . We need to solve  $2x = 4\lambda x^3$ ,  $2y = 4\lambda y^3$ ,  $2z = 4\lambda z^3$ ,  $x^4 + y^4 + z^4 = 8$ . For the first equation, we divide by  $x$  on both sides while keeping in mind that there might be solutions where  $x = 0$ , which we don't want to miss.

Now consider  $f = xy + 1$ ,  $g = x^2 + 4y^2$ . We need to solve  $y = \lambda 2x$ ,  $x = \lambda 8y$ ,  $x^2 + 4y^2 = 8$ . Note that if  $(x, y)$  is a solution,  $x, y$  must be both non-zero. It follows that, in this case, we can divide by  $2x$  in the first equation and  $8y$  in the second to conclude  $\frac{y}{2x} = \lambda = \frac{x}{8y}$  without losing anything.

In general, we can only divide by a variable under the assumption that it is non-zero. As the first example shows, to not miss any solutions, we must look at the case where that variable is zero as well.

- (4) Is  $\mathbb{R}^2$  closed and open? Are closed and open not mutually exclusive definitions?

**A:**  $\mathbb{R}^2$  is open and closed. In fact, the empty set and  $\mathbb{R}^2$  are the only subsets of  $\mathbb{R}^2$  that are open and closed. In general, a set can be open, closed, both or neither.

- (5) What is an intuitive way of thinking about line integrals of functions?

**Answer:** You can think of this as some sort of mean value theorem; in the sense that, recall that the average of  $f$  is defined to be *average of  $f$*   $= \frac{\int_C f dS}{\text{Length of } C}$ . Multiplying both sides by the length of  $C$ , we get that:  $\int_C f dS = \text{Length of } C \times (\text{average of } f)$ .

- (6) How do I approximate a scalar line integral given a contour plot?

**Answer:** You want to break the curve into small pieces along which the function is roughly constant, and then sum up the integral along those individual pieces. Say we want to compute  $\int_C g ds$ . Chop the curve  $C$  up into smaller pieces  $C_1, C_2, \dots, C_n$ . Use the contour plot to estimate the lengths  $l(C_1), \dots, l(C_n)$  and approximate values  $g_1, \dots, g_n$  of  $g$  on each piece  $C_i$ . Then the value of the line

integral will be approximately:

$$\int_C g \, ds \approx \sum_{i=1}^n g_i l(C_i).$$

See the solutions to the 2018 practice midterm, problem 5, for an example.

- (7) How do I find the path integral of a function along a curve with given parametric representation?  
**Answer:** Given a function  $f$  and a curve parametrized by  $(x(t), y(t))$  for  $a \leq t \leq b$ , the path integral is the one variable integral:

$$\int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt.$$

The three variable case is analogous.

- (8) How do I find the path integral of a vector field along a curve with given parametric representation?  
**Answer:** Given a vector field  $F(x, y) = (P(x, y), Q(x, y))$  and a curve parametrized by  $(x(t), y(t))$  for  $a \leq t \leq b$ , the path integral is the one variable integral:

$$\int_a^b (P(x(t), y(t)), Q(x(t), y(t))) \cdot (x'(t), y'(t)) dt.$$

- (9) Does the line integral over any closed curve equal zero?  
**Answer:** Not necessarily. If the vector field  $F$  you are integrating over is conservative, then the integral is zero. On the other hand, if  $F$  is not conservative, then it need not be zero. Consider the line integral of  $F(x, y) = (-y, x)$  over the curve  $(x(t), y(t)) = (\cos(t), \sin(t))$ .

- (10) How can you tell if a vector field is conservative based on its formula?  
**Answer:** If you're given a formula for a vector field  $\mathbf{F} = \langle f(x, y), g(x, y) \rangle$  defined on a simply connected domain, the condition for  $\mathbf{F}$  to be conservative is that  $f_y = g_x$ . Notice that you're taking the  $y$ -derivative of the  $x$ -coordinate function  $f$  and the  $x$ -derivative of the  $y$ -coordinate function  $g$ . A possibly confusing instance of this is that gradient vector fields are conservative. Since  $\nabla f = \langle f_x, f_y \rangle$ , for it to be conservative, we need  $f_{xy} = f_{yx}$ , which is precisely Clairaut's theorem.

- (11) If we have a vector field  $\mathbf{F} = \langle P, Q \rangle$  defined on some open and simply connected domain  $U \subset \mathbb{R}^2$  satisfying  $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$  then  $\mathbf{F}$  is conservative. Do we really need the region to be simply connected?  
**Answer:** Here's a counterexample: let  $U = \mathbb{R}^2 \setminus (0, 0)$  and define  $\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ . (Notice this would not be well defined on all of  $\mathbb{R}^2$ .) It turns out (and you should check!)

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$$

where  $C$  is the unit circle (with the usual parametrization). But the integral of a conservative vector field over a closed curve is always zero, so our  $\mathbf{F}$  is not conservative.

- (12) How can you tell if a vector field is conservative based on its plot?  
**Answer:** There are at least two possible approaches to this. One is to use the fact that the line integral of a conservative vector field along a closed path (one whose starting and ending points are the same) is always zero. If you can find such a path where the integral shouldn't be zero, then that vector field won't be conservative. To carry this out, first pick a simple path: a loop or even a rectangle. To eyeball the value of the line integral, note that in the line integral of a vector field, you're integrating the dot product of the vector field with the velocity vector of your parametrized curve—this dot product will be bigger when the vector field aligns with the direction of the curve, negative when they're close to being opposite, and close to zero if they're close to being orthogonal.

Another way is to use the definition of conservative:  $\mathbf{F} = \langle f(x, y), g(x, y) \rangle$  is conservative if  $f_y = g_x$ . Note that  $f$  is the  $x$ -coordinate and  $g$  is the  $y$ -coordinate, so this condition says that the change in the  $x$ -coordinates of the vectors as you move up should equal the change in the  $y$ -coordinates as you move right. In some problems (e.g. the third vector field in 2016 Q8), the vectors

don't change when you move in, say, the horizontal direction. This means that  $g_x = 0$ , so the vector field can't be conservative if  $f_y \neq 0$ .