1. Suppose $f(x,y)$ is a differentiable function with continuous second-order partial derivatives and values given in the table at right.

(x,y)	f	f_x	f_y	f_{xx}	f_{yy}	f_{xy}
(0,0)	0	0	0	2	1	4
(1,0)	5	0	−1	0	0	3
(0,1)	−3	0	0	−1	−2	1
(1,1)	0	−2	1	−1	5	6

(a) For each of the three points below, circle the best description of the point. Here "cannot determine" means "cannot determine from the information given." **(1 point each)**

(0, 0) is a:	local max	local min	saddle	not a critical point	cannot determine
(1, 0) is a:	local max	local min	saddle	not a critical point	cannot determine
(0, 1) is a:	local max	local min	saddle	not a critical point	cannot determine

(b) For $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$, circle the value of $D_{\mathbf{u}} f(1,1)$: $\frac{-7}{\sqrt{2}}$ $\quad \frac{-3}{\sqrt{2}}$ $\quad 0$ $\quad \frac{3}{\sqrt{2}}$ $\quad \frac{7}{\sqrt{2}}$ **(1 point)**

(c) The function f is **guaranteed** to take on an absolute maximum and minimum value on **exactly two** of the four sets below. Circle those two sets. **(1 point each)**

$\{(x,y) \mid x^2+y^2 \le 1\}$ closed & bdd $\quad \{(x,y) \mid y \ge 0\}$ not bounded $\quad \{(x,y) \mid x^2+y^2 < 2\}$ not closed $\quad \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ closed and bdd

Scratch Space

a) (0,0): critical as $f_x = f_y = 0$ $\quad D = \begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = -13$

$\Rightarrow$ saddle

(1,0): not crit as $f_y \neq 0$

(0,1): critical as $f_x = f_y = 0$ $\quad D = \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} = 1 > 0$

$f_{xx} < 0 \Rightarrow$ max

b) $D_{\vec{u}} f(1,1) = \nabla f(1,1) \cdot \vec{u}$

$= \langle -2, 1 \rangle \cdot \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle = \dfrac{-2\cdot 1 + 1\cdot(-1)}{\sqrt{2}} = \dfrac{-3}{\sqrt{2}}$

2. Find an equation for the tangent plane to the surface $xy^2 + yz^2 = x^2z + 1$ at the point $(0, 1, -1)$. **(4 points)**

Let $f(x,y,z) = xy^2 + yz^2 - x^2z$ so the surface is the level set $f = 1$. The normal is $\nabla f =$ $\langle y^2 - 2xz,\ 2xy + z^2,\ 2yz - x^2 \rangle$ at $(0,1,-1)$ which is $\langle 1, 1, -2 \rangle$. So eqn is

$1\cdot(x-0) + 1\cdot(y-1) + -2(z+1) = 0$ or

$x + y - 2z = 3$

Equation: $\boxed{1}\,x + \boxed{1}\,y + \boxed{-2}\,z = \boxed{3}$

3. *Use Lagrange multipliers* to find the maximum area of a rectangle in the first quadrant with two sides contained in the x and y axes and one corner on the line $2x + y = 12$. **You will not receive any credit for solving this problem unless you use Lagrange multipliers. (5 points)**

Maximize: $f(x,y) = \text{area} = xy$

Subject to: $g(x,y) = 2x + y = 12$

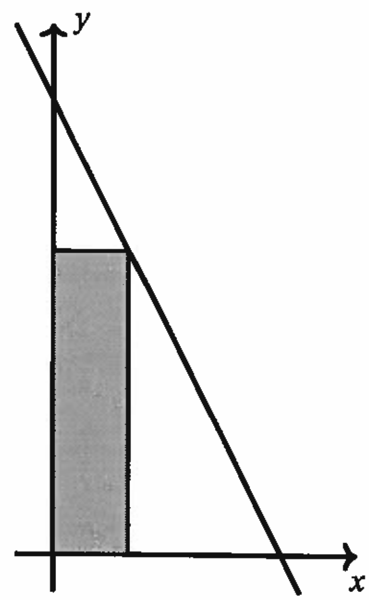
Then $\nabla f = \langle y, x \rangle = \lambda \nabla g = \lambda \langle 2, 1 \rangle$

gives $y = 2\lambda$ and $x = \lambda \implies$

$y = 2x$. Since $12 = 2x + y = 2x + 2x$

we learn $4x = 12 \implies x = 3$.

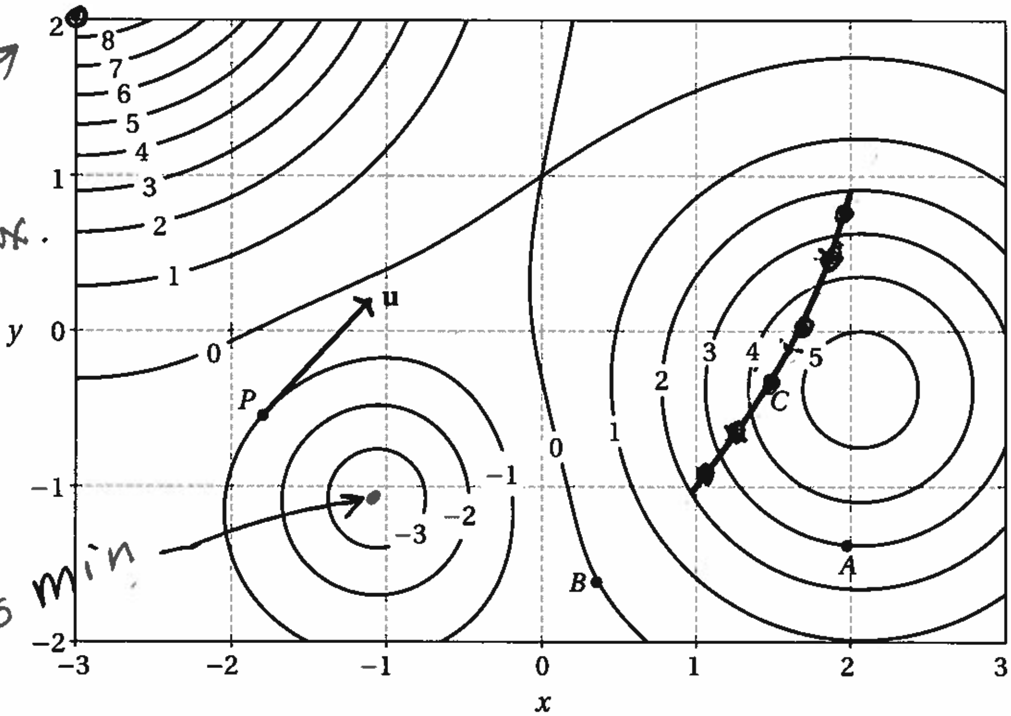
So only critical point is $(3, 6)$ and the max area is 18



Maximum Area = $\boxed{18}$

4. The contour map of a differentiable function $f: \mathbb{R}^2 \to \mathbb{R}$ is shown on the region $D = \{-3 \le x \le 3,\ -2 \le y \le 2\}$. Each level curve is labeled by the corresponding value of f. **(1 point each except where noted)**

Abs. max.



Abs min

C close to straight line from (1, -1) to (2, 1) so

$\text{len}(C) \approx \sqrt{5}$

≈ 2.2

(a) Find the gradient of f at $(0, 1)$: $\nabla f(0,1) = \langle 0\ ,\ 0 \rangle$

since f has a saddle at (0, 1).

(b) Circle the best estimate for the **absolute** maximum value of f on D:

DNE	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5	11.5

(c) Circle the best estimate for the **absolute** minimum value of f on D:

DNE	−11.5	−10.5	−9.5	−8.5	−7.5	−6.5	−5.5	−4.5	−3.5	−2.5	−1.5	−0.5

(d) Circle the correct relationship: $|\nabla f(A)| > |\nabla f(B)|$ $\quad |\nabla f(A)| = |\nabla f(B)|$ $\quad |\nabla f(A)| < |\nabla f(B)|$

(e) The directional derivative $D_{\mathbf{u}} f(P)$ is: negative / zero / positive

since $\vec{u}$ is tangent to the level set.

(f) For the curve C shown, circle the best estimate for $\int_C f\, ds$. **(2 points)**

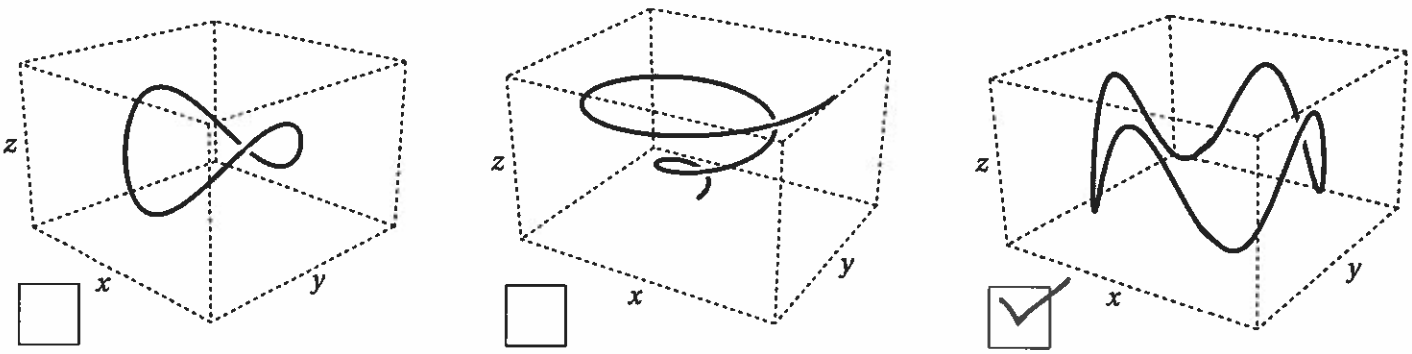
−16	−12	−8	−4	−2	0	2	4	8	12	16

Average on $C \approx \dfrac{2.5 + 3.5 + 4.5 + 4.5 + 3.5 + 2.5}{6}$

$= \dfrac{21}{6} = \dfrac{7}{2} = 3.5$ Length is $\approx \sqrt{5} = 2.2$ and so

$\int_C f\, ds = (\text{Len } C)\cdot(\text{Ave}) = 2.2 \times 3.5 = 7.7 \approx 8.$

5. Mark the box next to the curve that is parameterized by $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), \cos(4t)\rangle$ for $0 \le t \le 2\pi$. **(2 points)**



6. Let C be the curve parameterized by $\mathbf{r}(t) = \langle \cos t, \sin t, 2t\rangle$ for $0 \le t \le 2\pi$. Compute $\int_C z\, ds$. **(5 points)**

Have $\vec{r}'(t) = \langle -\sin t, \cos t, 2\rangle$ and so

Speed $= |\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 4} = \sqrt{5}$.

Now $\int_C z\, ds = \int_0^{2\pi} (2t) \underbrace{\sqrt{5}\, dt}_{ds} = \sqrt{5}\, t^2 \Big|_0^{2\pi}$

$= \sqrt{5}\, 4\pi^2$

$\boxed{\int_C z\, ds = 4\sqrt{5}\,\pi^2}$

Scratch Space

7. For all parts of this problem, the curve C is the oriented straight line segment from $(0, 0)$ to $(1, 1)$.

(a) For $\mathbf{F} = \langle x, -y^2 \rangle$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$. **(4 points)** Use $\vec{r}(t) = \langle t, t \rangle$ for $0 \le t \le 1$

so $\vec{r}\,'(t) = \langle 1, 1 \rangle$. Then $\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$$= \int_0^1 \langle t, -t^2 \rangle \cdot \langle 1, 1 \rangle \, dt$$

$$= \int_0^1 t - t^2 \, dt = \frac{t^2}{2} - \frac{t^3}{3} \Big|_{t=0}^{1} = 1/2 - 1/3 = 1/6$$

$\int_C \mathbf{F} \cdot d\mathbf{r} =$ $\boxed{1/6}$

(b) For $f(x, y) = \cos\left(\frac{\pi}{2} xy\right) e^{xy}$, compute $\int_C \nabla f \cdot d\mathbf{r}$. **(2 points)** By the F.T.L.I.

have $\int_C \nabla f \cdot d\vec{r} = f(1,1) - f(0,0)$

$$= \cos(\pi/2)\, e^1 - \cos(0) \cdot e^0$$

$$= 0 - 1$$

$\int_C \nabla f \cdot d\mathbf{r} =$ $\boxed{-1}$

(c) Circle the best estimate for $\frac{1}{\sqrt{2}} \int_C e^{\frac{xy}{200}} \cos\left(\frac{xy}{200}\right) ds$. **(1 point)**

−3	−2	−1	0	1	2	3

Scratch Space

c) Along C, the fn $g = e^{xy/200} \cos \frac{xy}{200}$ is very close to 1, so $\frac{1}{\sqrt{2}} \int_C g \, ds =$ Average of g on C ≈ 1.

8. The vector field $\mathbf{F} = \langle 2xy + 2x + y^2,\ 2xy + 2y + x^2 \rangle$ is conservative. Find a potential function f for $\mathbf{F}$ (a function with $\nabla f = \mathbf{F}$). **No partial credit - you can check your answer! (2 points)**

First, $f = \int f_x\, dx = x^2 y + x^2 + y^2 x + C(y)$. Then

$$\frac{\partial}{\partial y} f = x^2 + 2yx + \frac{\partial C}{\partial y} \Rightarrow \frac{\partial C}{\partial y} = 2y \Rightarrow C = y^2$$

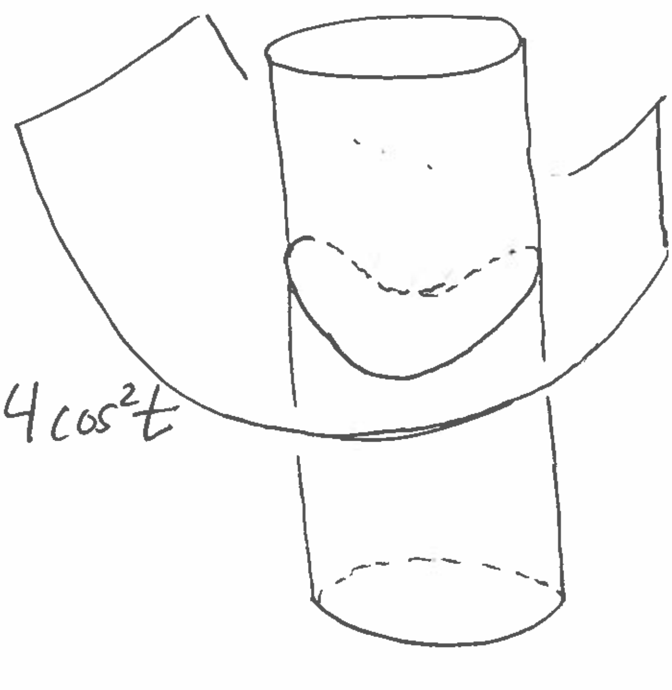
So

$f(x, y) = \boxed{x^2 y + x^2 + y^2 x + y^2}$

9. Find a vector function $\mathbf{r}(t)$ that parameterizes the intersection of the circular cylinder $x^2 + y^2 = 4$ with the parabolic cylinder $z = x^2$, traversing the curve exactly once. Be sure to specify the domain of your parameterization. **(4 points)**

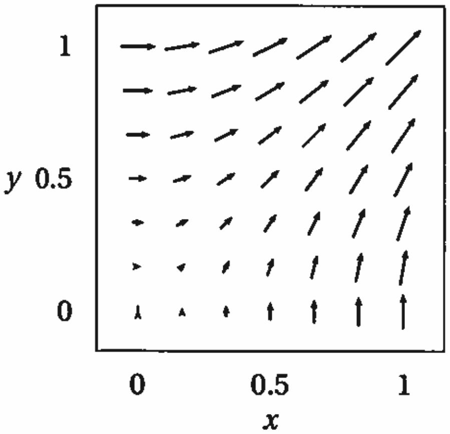
From above, the curve looks like a circle of radius 2, so take $x = 2\cos t$, $y = 2\sin t$

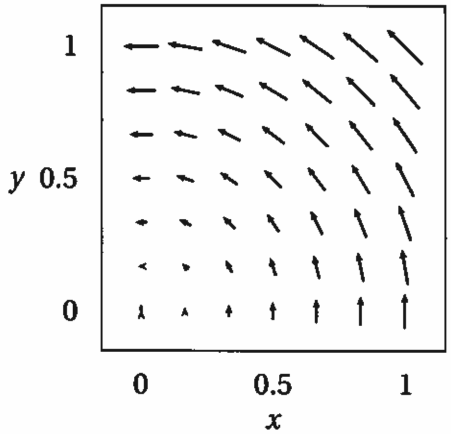
and as $z = x^2$ we get $z = 4\cos^2 t$

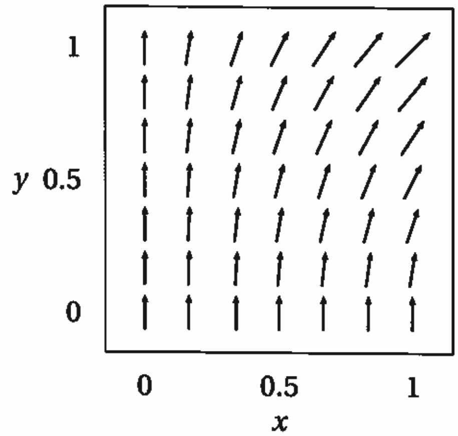


$\mathbf{r}(t) = \boxed{\langle 2\cos t, 2\sin t, 4\cos^2 t \rangle}$ for $\boxed{0} \le t \le \boxed{2\pi}$

10. (a) Match each vector field $\mathbf{F} = \langle -y, x \rangle$ and $\mathbf{G} = \langle xy, 1 \rangle$ on $\mathbb{R}^2$ with its pictures below and label the corresponding boxes. **(1 point each)**

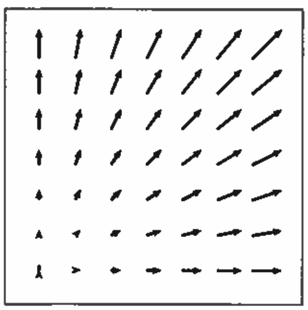


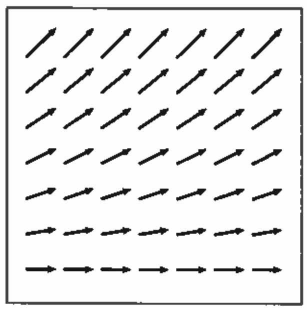


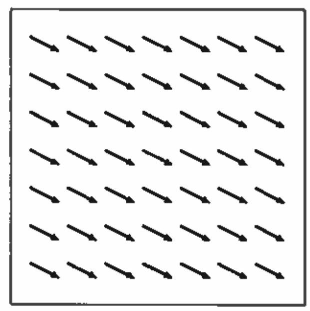


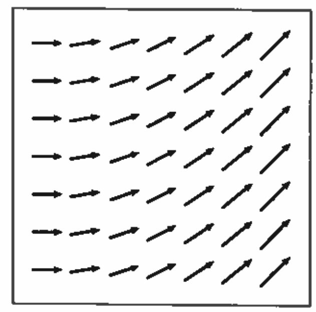
	$\vec{F}$	$\vec{G}$

(b) Exactly one of the vector fields below is **not** conservative; mark the box below it. **(2 points)**





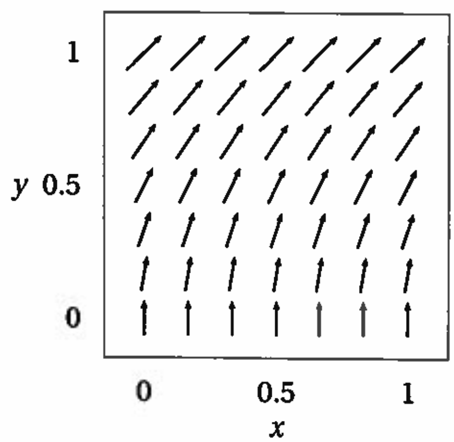




			X

(c) Circle the formula for $\mathbf{r}(t)$ that gives a streamline (also called a flowline or an integral curve) for the vector field shown at right. **(2 points)**

$\langle t/2, t \rangle$	$\langle t^2/2, t \rangle$	$\langle t, t^2/2 \rangle$	$\langle 1/2, t \rangle$



(d) Suppose $\mathbf{E}$ is a vector field on $\mathbb{R}^2$ where the streamline through $(1,0)$ is the ellipse $x^2 + y^2/4 = 1$. Is $\mathbf{E}$ conservative? **(1 point)**

yes	no	cannot determine