1. Find the maximum and minimum value of the function $f(x,y,z) = 3x+y$ on the ellipsoid $3x^2+2y^2+z^2 = 14$. **(5 points)**

Let $g(x,y,z) = 3x^2+2y^2+z^2-14$. This is our constraint function

$f(x,y,z) = 3x+y$. This is the function we want to maximize/minimize

$\nabla f = (3,1,0)$, $\nabla g = (6x, 4y, 2z)$ [note: $\nabla g = 0$ only at $(0,0,0)$ but this point is not on the ellipsoid.

Lagrange Multipliers: we want to solve the System: $\nabla f = \lambda \nabla g$ & $g = 0$

eq's: ① $3 = 6\lambda x$

② $1 = 4\lambda y$

③ $0 = 2\lambda z$

④ $0 = 3x^2 + 2y^2 + z^2 - 14$

by eqⁿ ①, $\lambda \neq 0$, so eqⁿ ③ gives $z = 0$.

$6\lambda x = 3 = 12\lambda y$, Since $\lambda \neq 0$, $x = 2y$ Plug $(2y, y, 0)$ into ④:

$0 = 3(2y)^2 + 2y^2 + (0)^2 - 14$ i.e. $14y^2 = 14$ So $y = \pm 1$

the two critical pts are $(2,1,0)$ & $(-2,-1,0)$

$f(2,1,0) = 7$

$f(-2,-1,0) = -7$

Maximum value = 7

Minimum value = −7

2. Find the length of the curve C parameterized by $\mathbf{r}(t) = \langle 2t, \cos t, \sin t\rangle$ for $0 \le t \le 5\pi$. **(4 points)**

$$L = \int_0^{5\pi} |r'(t)|\, dt$$

$$r'(t) = \langle 2, -\sin t, \cos t\rangle$$

$$|r'(t)| = \sqrt{2^2 + (-\sin t)^2 + (\cos t)^2} = \sqrt{5}$$

$$= \int_0^{5\pi} \sqrt{5}\, dt = 5\sqrt{5}\pi$$

Length = $5\sqrt{5}\,\pi$

3. Parameterize the curve given by the intersection of the paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$. Specify the domain (the values of the parameter t) so that the function traces the curve exactly once. **(4 points)**

The intersection is pts (x, y, z) satisfying

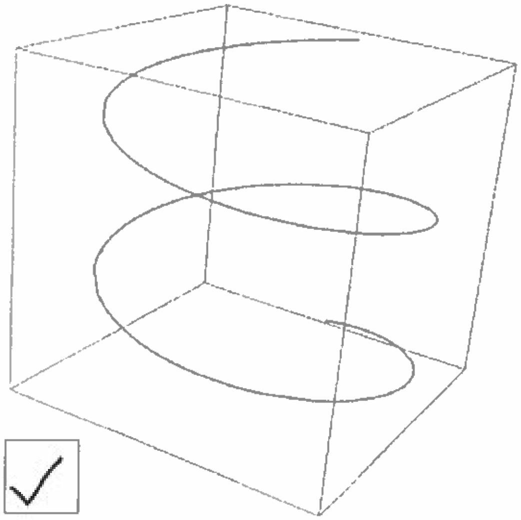
$y = x^2$ & $z = 4x^2 + y^2$

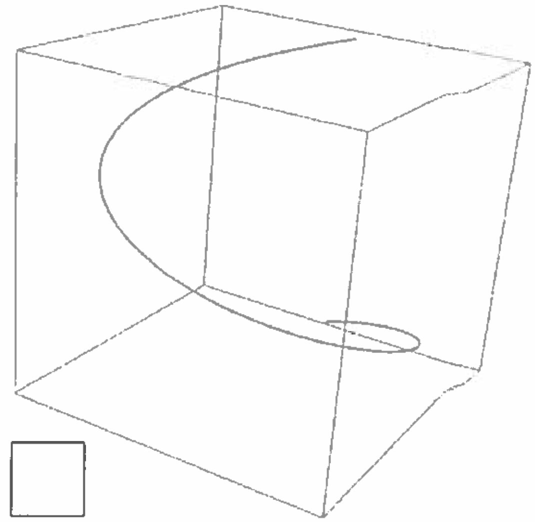
i.e. $y = x^2$ & $z = 4x^2 + (x^2)^2 = 4x^2 + x^4$

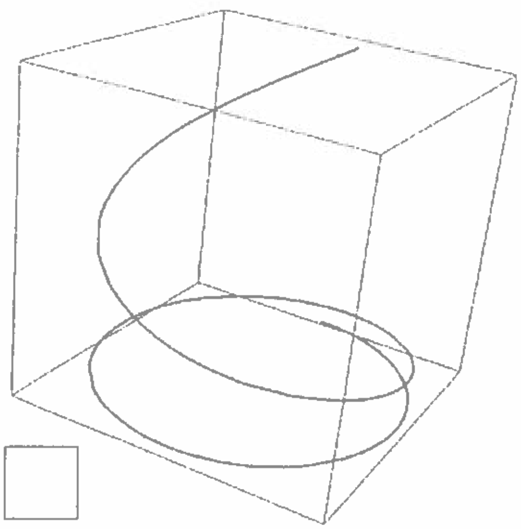
So we can paramdrize this as $(t, t^2, 4t^2 + t^4)$ where $t \in \mathbb{R}$

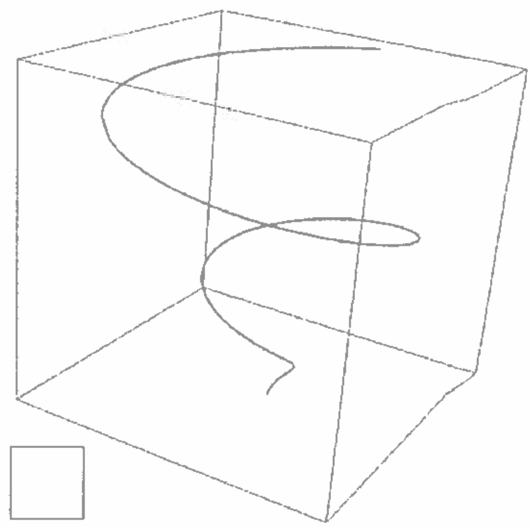
$\mathbf{r}(t) = \langle\ t\ ,\ t^2\ ,\ 4t^2 + t^4\ \rangle$ for t in $\mathbb{R}$

4. Let C be the curve parameterized by $\mathbf{r}(t) = \langle \sin(t^2), \cos(t^2), t^2 \rangle$ for $0 \le t \le 2\sqrt{\pi}$. Check the box below the picture of the curve C. **(2 points)**









5. Consider the following domains (subsets) in $\mathbb{R}^2$. For each, circle the characteristics that accurately describe the set; **circle zero, one, or both options**, as appropriate. **(3 points)**

(a) $D_1 = \{(x, y) \mid y^2 \ge x^2 + 1\}$. D_1 is | simply connected | closed |

(b) $D_2 = \{(x, y) \mid 1 < x^2 + y^2 \le 2\}$. D_2 is | open | bounded |

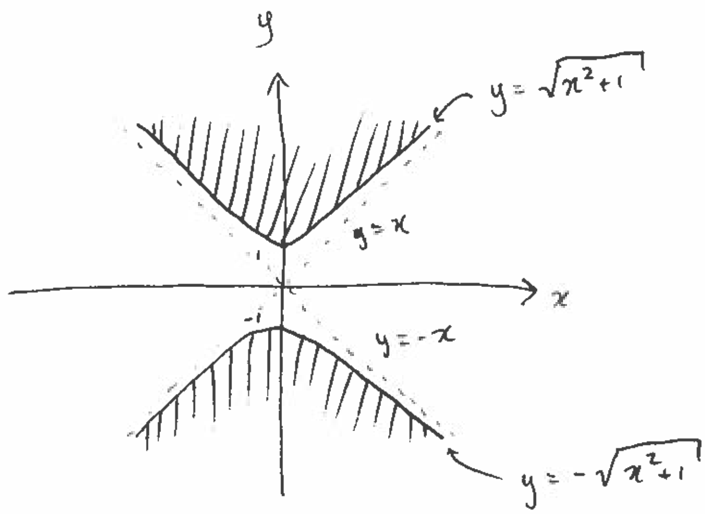
(c) $D_3 = \{(x, y) \mid (x, y) \ne (0, 0)\}$. D_3 is | simply connected | bounded |

4) the image of $\underline{r}(t)$ is the same as the image of

$\underline{s}(t) = \langle \sin(t), \cos(t), t \rangle$ for $0 \leq t \leq 4\pi$

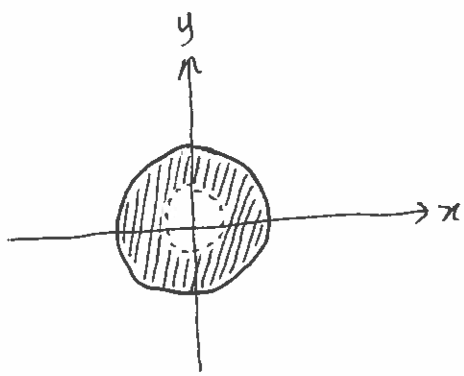
the answer is now clear.

5) a) D_1



- not even connected so certainly not simply connected
- It contains its boundary, in this case the curves given by $y=\sqrt{x^2+1}$ and $y=-\sqrt{x^2+1}$, so its closed

b) D_2



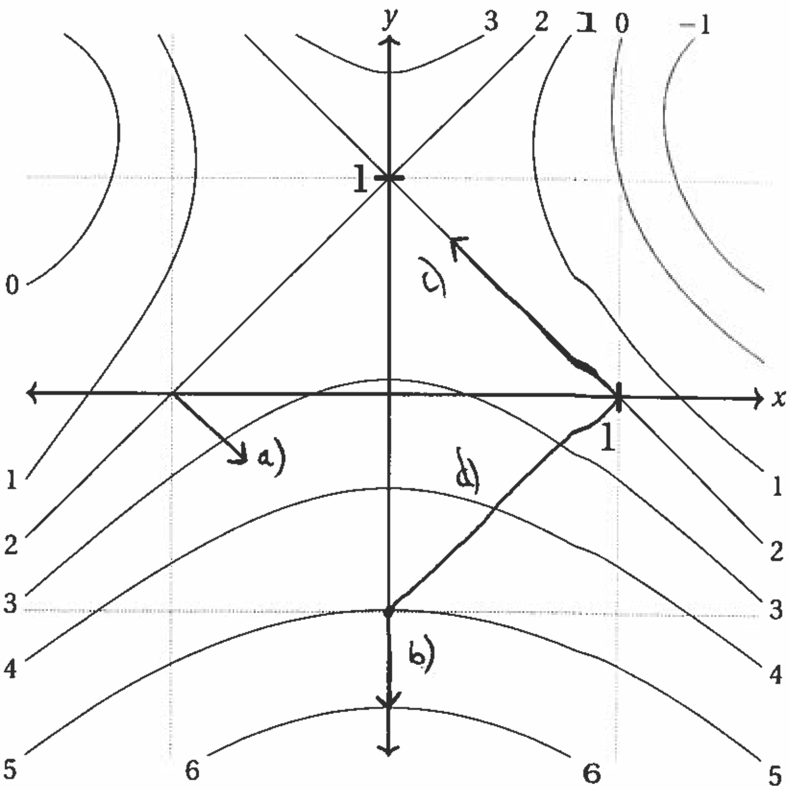
- Its not open, if you take a small ball around the point $(2,0)$ it will contain points not in D_2
- Its clearly bounded.

c) D_3 - This has a 'hole' at $(0,0)$ so is not simply connected.

- Its clearly not bounded.

6. The contour map of a differentiable function $f(x, y)$ is shown below.

Circle the *best response* for each of parts (a) - (d) below.



(a) **(2 points)** $\nabla f(-1,0) =$

$\langle 0,1\rangle$	$\langle -1,2\rangle$	$\langle 1,-1\rangle$	$\langle 2,-2\rangle$
$\langle 1,0\rangle$	$\langle 1,-2\rangle$	$\langle 0,-2\rangle$	$\langle -2,2\rangle$

(b) **(2 points)** The maximum rate of change of f at $(0,-1)$ is

1	2	3	4	5	6

(c) **(2 points)** $D_{\left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle} f(1,0) =$

-2	-1	0	1	2

(d) **(2 points)** If C is the straight line segment from $(0,-1)$ to $(1,0)$, then

$\frac{1}{\sqrt{2}}\int_C f\,ds =$

-5	$-\frac{7}{2}$	$-\frac{3}{2}$	0	$\frac{1}{2}$	2	$\frac{7}{2}$	5

Scratch Space

a) the direction in which f increases fastest is $\langle 1,-1\rangle$, so $\nabla f(-1,0) = \lambda\langle 1,-1\rangle$

So either $\nabla f(-1,0)$ is $\langle 1,-1\rangle$ or $\langle 2,-2\rangle$. Let's think about the directional derivative in the direction $\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle$. From the picture it takes roughly $\frac{1}{3}$ units length for the function to increase from 1 to 2, so we should expect the directional derivative to be ≈ 3. $\lambda\langle 1,-1\rangle \cdot \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle = \lambda\sqrt{2}$

for $\lambda\sqrt{2} \approx 3$ we would want $\lambda \approx 2$.

b) the direction of maximum rate of change is $\langle 0, -1 \rangle$

The function increases from 4 to 6 in roughly 1 unit length (from $(0, -\frac{1}{2})$ to $(0, -\frac{3}{2}$

So the maximum rate of change is 2.

c) The vector lies along a level curve, So the directional derivative is 0.

d) $\frac{1}{\sqrt{2}} \int_C f\,ds = \frac{1}{\sqrt{2}} \cdot$ length of C $\cdot$ 'avg. value of f on C'.

The max value of f on C is 5, the minimum is 2 So the average is $\approx \frac{5+2}{2} = \frac{7}{2}$

the length of C is $\sqrt{2}$, So $\frac{1}{\sqrt{2}} \int_C f\,ds \approx \frac{1}{\sqrt{2}} \cdot \sqrt{2} \cdot \frac{7}{2} = \frac{7}{2}$

7. Three continuous vector fields, **F**, **G**, **H** on the plane are plotted below in the region with $-1 < x < 1$ and $-1 < y < 1$.

Circle the *best response* for each of the following.

(a) **(2 points)** Which vector field is ∇f, where $f(x, y) = xy$?

F	G	H

(b) **(4 points)** A particle moves along a straight line from $(-1, -1)$ to $(1, 1)$. If the vector fields represent force fields, then:

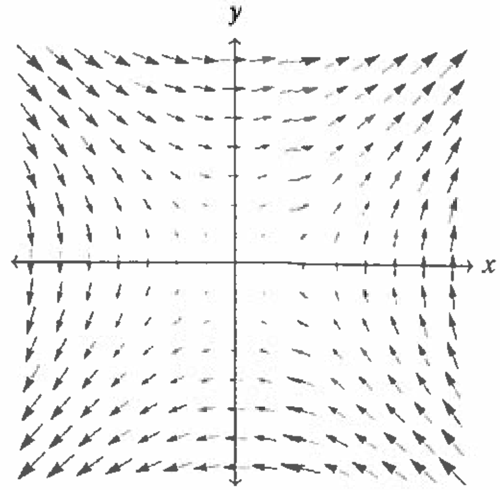
The work done by **G** on the particle is...

positive	negative	zero

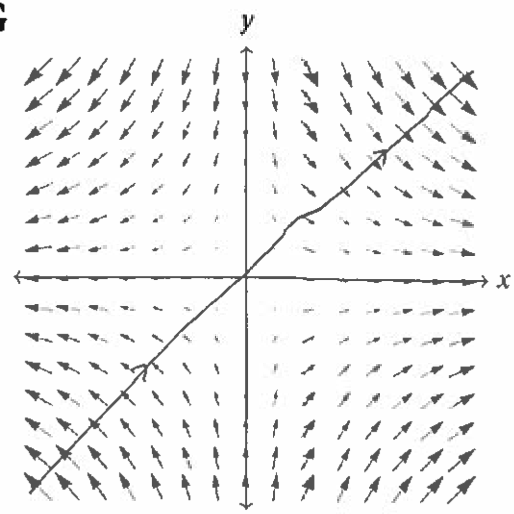
The work done by **H** on the particle is...

positive	negative	zero

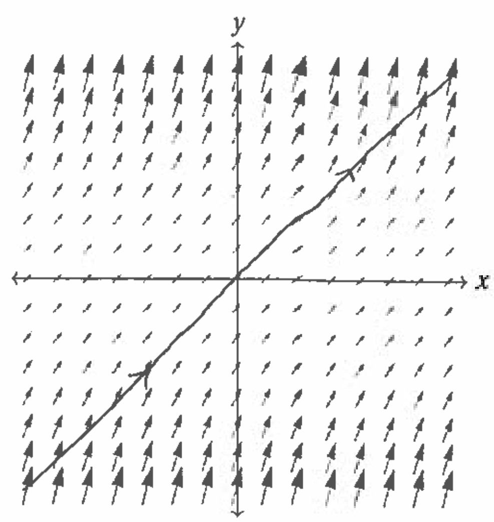
F



G



H



a) $\nabla f(x, y) = (y, x)$

So for example $\nabla f(1,0) = (0,1)$ which can only be **F**

b) **G**: at every point on the line the direction of the vector field is orthogonal to the direction vector of the line hence the work done is 0.

b) **H**: at every point on the line the direction of the vector field makes an angle $\cancel{\frac{\pi}{2}}$ $< \frac{\pi}{2}$ w/ the direction vector of the line hence the work done is +ve.

8. Consider the vector field $\mathbf{F}(x, y) = \langle \frac{1}{y}e^{\frac{x}{y}}, -\frac{x}{y^2}e^{\frac{x}{y}} + 2y \rangle$. Let C be the straight line segment from $P = (1, 1)$ to $Q = (2, 2)$ parametrized by

$$\mathbf{r}(t) = \langle t, t \rangle, \quad \text{for} \quad 1 \le t \le 2.$$

(a) Use the definition of the line integral of a vector field along a curve to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ directly, using the above parametrization. **(No credit will be given for computations using any other method.)** **(5 points)**

$$F(r(t)) = \langle \tfrac{1}{t}e, \tfrac{-1}{t}e + 2t \rangle, \quad r'(t) = \langle 1, 1 \rangle$$

$$F(r(t)) \cdot r'(t) = \tfrac{1}{t}e - \tfrac{1}{t}e + 2t = 2t$$

$$\int_C F \cdot dr = \int_1^2 2t \, dt = [t^2]_1^2 = 4 - 1 = 3$$

$\int_C \mathbf{F} \cdot d\mathbf{r} =$ $\boxed{3}$

(b) The vector field $\mathbf{F}$ is conservative. Find a function f such that $\nabla f = \mathbf{F}$. **(2 points)**

$f_x = \frac{1}{y}e^{\frac{x}{y}}$ So $f = e^{\frac{x}{y}} + g(y)$

then need $f_y = \frac{-x}{y^2}e^{\frac{x}{y}} + 2y$ but have $f_y = \frac{-x}{y^2}e^{\frac{x}{y}} + g'(y)$

So any $g(y)$ s.t. $g'(y) = 2y$ will work, e.g. $g(y) = y^2$ $f(x, y) =$ $\boxed{e^{\frac{x}{y}} + y^2}$

(c) Use your work from part (b) to check your result from part (a). (Show your work and explain your method. Note: If you were not able to solve part (a) or part (b), explain how you *could* check your answer assuming that you had found a number N in part (a) and a function f in part (b).) **(2 points)**

Fundamental theorem of Line integrals:

Q is the endpoint of C

P is the initial point of C

for $F = \nabla f$ we have $\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(Q) - f(P)$

in our case $Q = (2,2)$, $P = (1,1)$

so $f(2,2) = e + 4$, $f(1,1) = e + 1$

then $\int_C F \cdot dr = f(2,2) - f(1,1) = 3.$

8. Consider the vector field $\mathbf{F}(x,y) = \langle \frac{1}{y}e^{\frac{x}{y}}, -\frac{x}{y^2}e^{\frac{x}{y}} + 2y \rangle$. Let C be the straight line segment from $P = (1,1)$ to $Q = (2,2)$ parametrized by

$$\mathbf{r}(t) = \langle t, t \rangle, \quad \text{for} \quad 1 \le t \le 2.$$

(a) Use the definition of the line integral of a vector field along a curve to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ directly, using the above parametrization. **(No credit will be given for computations using any other method.)** **(5 points)**

$$\int_C F \cdot dr = \int_1^2 \langle \tfrac{1}{t}e^1, -\tfrac{1}{t}e^1 + 2t \rangle \cdot \langle 1, 1 \rangle \, dt$$

$$= \int_1^2 \cancel{\frac{e}{t}} - \cancel{\frac{e}{t}} + 2t \, dt = t^2 \Big|_1^2 = 4 - 1 = 3$$

$\int_C \mathbf{F} \cdot d\mathbf{r} =$ $\boxed{3}$

(b) The vector field $\mathbf{F}$ is conservative. Find a function f such that $\nabla f = \mathbf{F}$. **(2 points)**

$f_x = \frac{1}{y} e^{\frac{x}{y}}$ so $f = e^{\frac{x}{y}} + g(y)$

$f_y = \frac{-x}{y^2} e^{\frac{x}{y}} + g_y$ so $g_y = 2y$ so $g = y^2 + C$ say $C = 0$

$f(x,y) =$ $\boxed{e^{\frac{x}{y}} + y^2}$

(c) Use your work from part (b) to check your result from part (a). (Show your work and explain your method. Note: If you were not able to solve part (a) or part (b), explain how you *could* check your answer assuming that you had found a number N in part (a) and a function f in part (b).) **(2 points)**

$F = \nabla f$ so $\int F \cdot dr = f(2,2) - f(1,1) = (e + 4) - (e + 1)$

$= 3$

9. Suppose that $f(x,y)$ is a differentiable function with continuous second order partial derivatives and values given by the table below. **(5 points)**

(x,y)	$f(x,y)$	$f_x(x,y)$	$f_y(x,y)$	$f_{xx}(x,y)$	$f_{yy}(x,y)$	$f_{xy}(x,y)$
$(2,1)$	0	0	-1	2	3	1
$(0,1)$	-1	0	0	-2	-2	-2
$(1,0)$	2	0	0	2	1	1

For each of the given points, circle the best description of the point.

$(2,1)$	not critical	local minimum	local maximum	saddle point	undetermined
$(0,1)$	not critical	local minimum	local maximum	saddle point	undetermined
$(1,0)$	not critical	local minimum	local maximum	saddle point	undetermined

$(2,1)$: $f_y(2,1) \neq 0$ so this is not a critical point.

$(0,1)$: both f_x & f_y are zero so it's a critical pt but $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & -2 \\ -2 & -2 \end{vmatrix} = 0$ so undetermin

$(1,0)$: $f_x = f_y = 0$ so critical pt. $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1 > 0$ & $f_{xx} > 0$ So local minimum.

10. Let D be the set of all points (x,y) in $\mathbb{R}^2$ except for $(0,0)$. In each part below, indicate whether a continuous vector field with domain D and the property described is *necessarily conservative* or *not necessarily conservative*. (If the vector field is never conservative, circle *not necessarily conservative.*) **(4 points)**

(a) $\mathbf{F}_1(x,y) = \langle -y, x\rangle$.

$\mathbf{F}_1$ is

necessarily conservative	not necessarily conservative

(b) $\mathbf{F}_2$ has the path independence property; that is, the line integral $\int_C \mathbf{F}_2 \cdot d\mathbf{r}$ is independent of path in D.

$\mathbf{F}_2$ is

necessarily conservative	not necessarily conservative

(c) $\mathbf{F}_3 = \langle P, Q\rangle$ where $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ over D.

$\mathbf{F}_3$ is

necessarily conservative	not necessarily conservative

(d) $\mathbf{F}_4$ has the property that for the unit circle $C_1 = \{x^2 + y^2 = 1\}$, $\int_{C_1} \mathbf{F}_4 \cdot d\mathbf{r} = 0$.

$\mathbf{F}_4$ is

necessarily conservative	not necessarily conservative

10 a) If $F_1 = \nabla f$ then $f_x = -y$ so $f = -xy + g(y)$

but then $f_y = -x + g'(y)$ there is no way we can make this equal to x ($g'(y)$ is a function of y) So there is no f s.t. $F = \nabla f$.

b) This statement is equivalent to F_2 being conservative.

c) IF D was simply connected, then this statement would allow us to conclude F_3 was conservative, but D is not simply connected so we can't make this conclusion.

counter-example: $F = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$ [notice that this would not make sense at the point $(0,0)$.

d) If we had that the integral over all closed paths was zero, this would be enough to conclude that F_4 was conservative. We only have that the integral over the unit circle is zero.

counter-example: $F = \left((x^2+y^2-1)\frac{-y}{x^2+y^2}, (x^2+y^2-1)\frac{x}{x^2+y^2} \right)$

Forcing the vector field to be $\langle 0,0 \rangle$ on C_1.