Math 518: HW 8 due Wednesday, October 29, 2014.

1. Problem 11–4 of Lee on pages 299–300. You may assume that M has no boundary.

2. Let $G = \mathrm{GL}_n\mathbb{R}$. Define n^2 covector fields σ_{ij} on G for $1 \le i, j \le n$ as follows. At a point $X = (x_{ij})$ in G, define

$$\sigma_{ij} = \sum_{k=1}^{n} y_{ik}\, dx_{kj} \quad \text{where } Y = (y_{ik}) \text{ is the inverse matrix to } X.$$

   Since taking the inverse is a smooth function on the Lie group G, the σ_{ij} are in $\Omega^1(M)$.

   (a) Prove that each σ_{ij} is left-invariant. That is, for every $A \in G$, prove that $(L_A)^*(\sigma_{ij}) = \sigma_{ij}$.

   (b) Prove that at every $A \in G$, the 1-forms σ_{ij} form a basis for T_A^*G.

   (c) Now take $n = 2$ and consider the circle $C \subset G$ which is the image of $\gamma(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ for $t \in [0, 2\pi]$. Calculate each of the integrals $\int_C \sigma_{ij} = \int_0^{2\pi} \gamma^*(\sigma_{ij})$.

3. Problem 11–13 of Lee on Page 301.

4. Let g be the round Riemannian metric on S^2, that is, $g_p\colon T_pS^2 \times T_pS^2 \to \mathbb{R}$ is the restriction to $T_pS^2 \subset T_p\mathbb{R}^3$ of the usual dot product on $\mathbb{R}^3$, where as always $S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\}$.

   Compute the coordinate form of g with respect to each of the following charts. That is, for a chart (U, ϕ) compute the four functions $g_{ij}\colon \phi(U) \to \mathbb{R}$ so that

$$(\phi_* g)_{(x_1,x_2)} = \sum_{i,j=1}^{2} g_{ij}(x_1, x_2) \cdot (dx_i \otimes dx_j)$$

   (a) Sterographic projection away from $(0, 0, 1)$.

   (b) Projection of the open upper hemisphere of S^2 onto the xy-plane.

   In which of these coordinates do the angles implicitly assigned by $\phi_* g$ agree with the ordinary Euclidean angles on $\mathbb{R}^2$?

5. Problem 13–21 of Lee on page 347.

6. Problem 12–7 of Lee on page 325.