Math 518, Lecture 1: Introduction

Smooth manifolds: "Spaces locally like $\mathbb{R}^n$ on which we can do calculus."

Ex: $\mathbb{R}^n$, circle $S^1 =$ , surfaces in $\mathbb{R}^3$

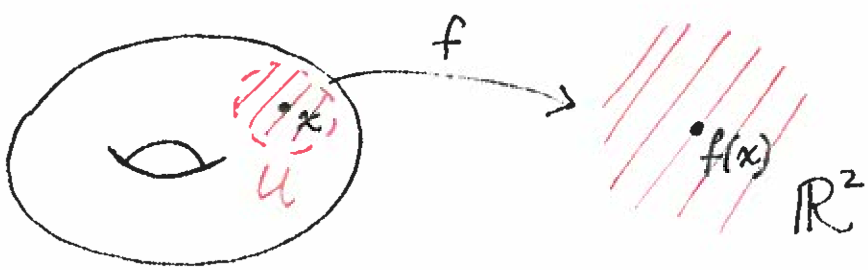
S^2:  

$S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}, \ldots$

[Calculus: velocities, areas, ODE, PDE, ... ; arise in many contexts.]

[This is not the only kind of manifold...]

Def: A topological/metric space X is locally Euclidean of dimension n if every $x \in X$ is contained in some open set $U \subseteq X$ where $U \cong \mathbb{R}^n$.
homeomorphic



f continuous bijection with continuous inverse.

Def: A topological n-manifold is a "reasonable" topological space which is locally Euclidean of dimension n.

"reasonable": (a) A metric space X with a countable dense set D. $[\forall x \in X \text{ and } \varepsilon > 0,\ B_\varepsilon(x) \cap D \neq \phi]$

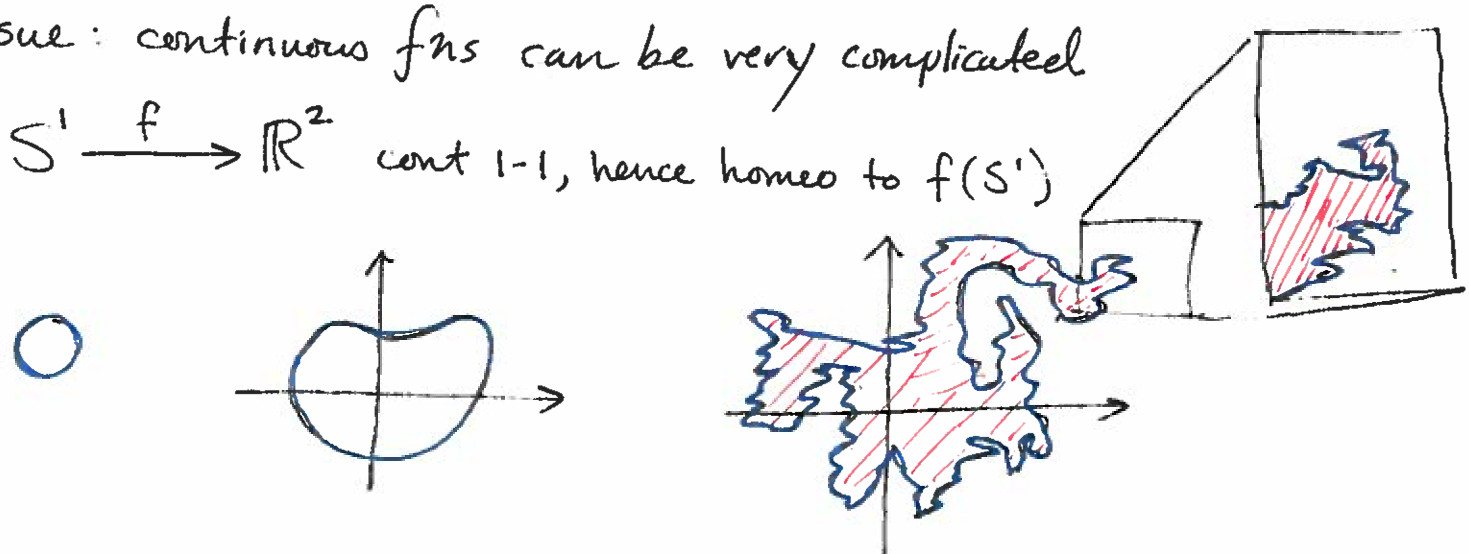
(b) A topological space X which is Hausdorff and has a countable basis (2^{nd} countable)

[(a) $\Rightarrow$ (b) and if X is locally Euclid then (a) $\Leftrightarrow$ (b)]

[A smooth mfld is top mfld + some add'l structure]

Issue: continuous fns can be very complicated

$S^1 \xrightarrow{f} \mathbb{R}^2$ cont 1-1, hence homeo to $f(S^1)$



<u>Schönflies Thm</u>: $\exists$ a homeo h of $\mathbb{R}^2$ with $h(f(S^1))$ = round circle; in particular, one component of $\mathbb{R}^2 \setminus f(S^1) \cong \mathbb{R}^2$.

False for embeddings of $S^2 \hookrightarrow \mathbb{R}^3$!

(Alexander horned sphere)

<u>Def:</u> A function $f : (U \overset{\text{open}}{\subseteq} \mathbb{R}^n) \longrightarrow \mathbb{R}$ is smooth (C^∞) if all partial derivatives of all orders exist everywhere on U.

<u>Ex:</u> f a polynomial, e.g. $f(x,y,z) = x^2y + z^4 + yzx + 2$

What is the geometric/physical meaning of $\dfrac{\partial^{10} f}{\partial^2 x \, \partial^5 y \, \partial^3 z}$?

<u>Better Def:</u> f is smooth if $\forall d \in \mathbb{Z}_{\geq 0}$ and $x_0 \in U$ there is a polynomial p of degree d where

$$f(x_0 + h) = p(h) + O(|h|^{d+1})$$

That is, there are constants $C, \varepsilon > 0$ such that

$$|f(x_0+h) - p(h)| \leq C|h|^{d+1} \quad \text{for all } |h| < \varepsilon.$$

<u>Ex:</u> $f(x,y) = (2\sin y + \cos x)\, e^{(x + \sin y)}$

Near $(0,0)$: $f(x,y) \approx 1$

$\frac{\partial f}{\partial y}(0,0)$

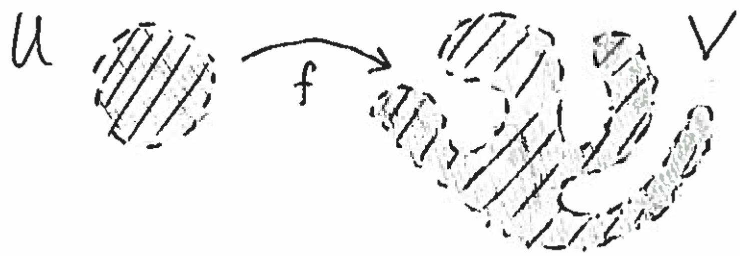
$\approx 1 + x + 3y$

$\frac{\partial f}{\partial x \partial y}(0,0)$ $\quad \frac{1}{2}\frac{\partial f}{\partial y^2}(0,0)$

$\approx 1 + x + 3y + 3xy + \frac{5}{2}y^2$

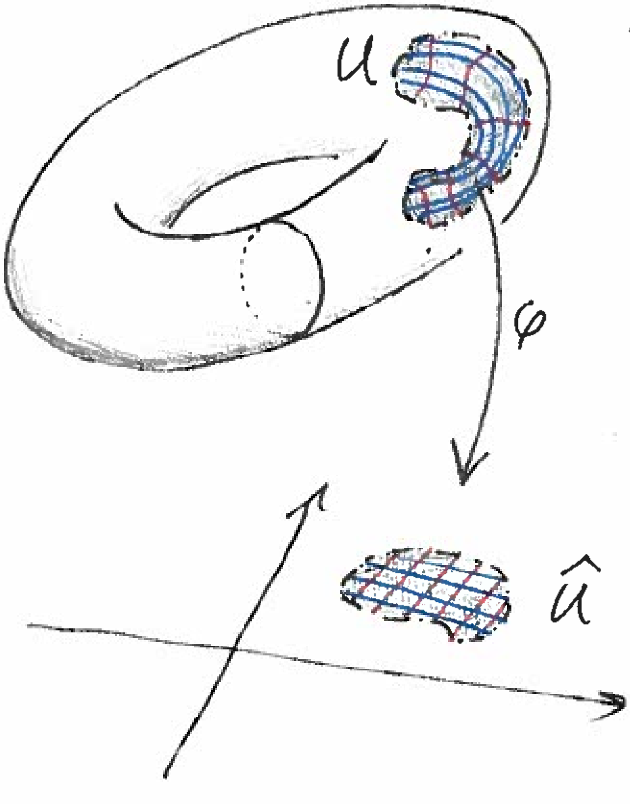
A map $f : (U \overset{open}{\subseteq} \mathbb{R}^n) \longrightarrow \mathbb{R}^m$ is smooth if each coordinate function $f_k : U \to \mathbb{R}$ is smooth.

A bijection f between open subsets U and V of $\mathbb{R}^n$ is a <u>diffeomorphism</u> if f and f^{-1} are <u>both</u> smooth.



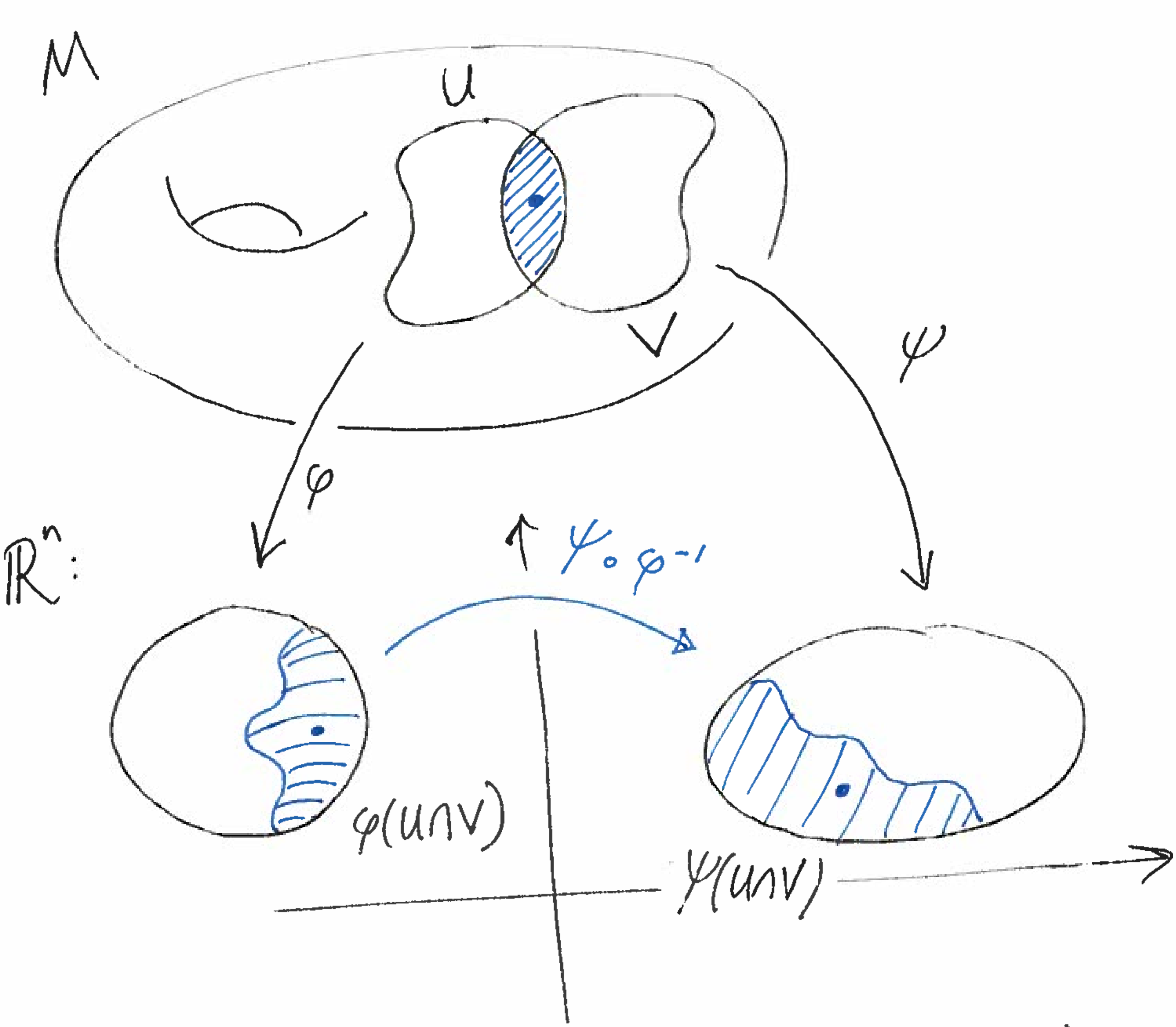
M a top n-mfld. A <u>coordinate chart</u> is a homeomorphism

$$\varphi : (U^{open} \subseteq M) \longrightarrow (\hat{U}^{open} \subseteq \mathbb{R}^n)$$



Two charts (U, φ) and (V, ψ) are <u>smoothly compatible</u> if the <u>transition map</u> is a diffeomorphism.

see next page.

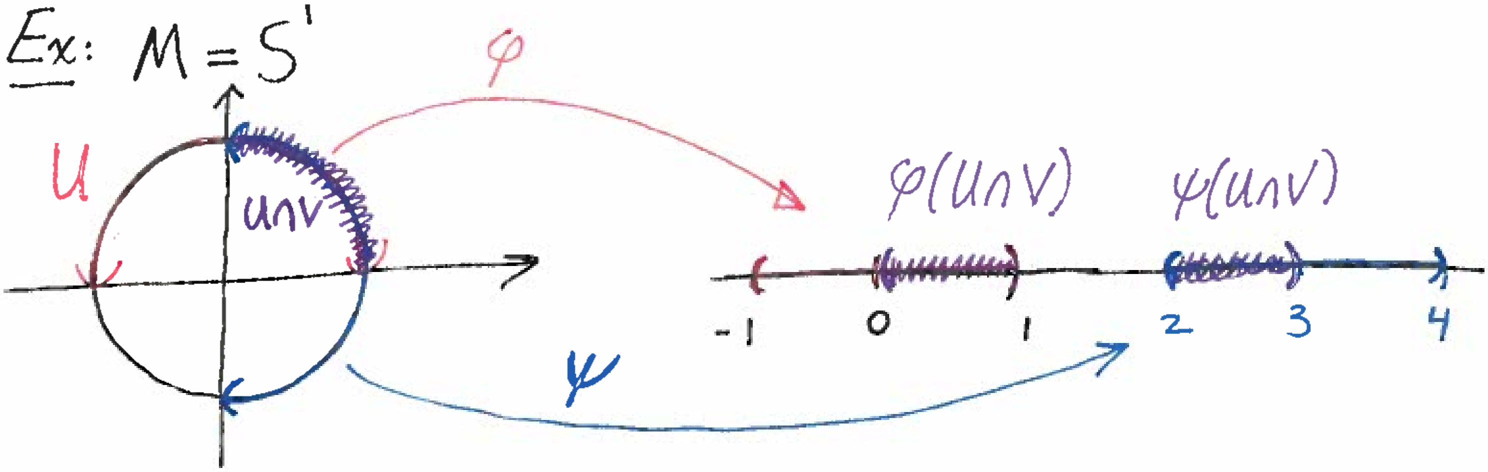


Transition map: $\psi\circ\varphi^{-1} : \varphi(U\cap V) \longrightarrow \psi(U\cap V)$, a homeomorphism.

A smooth atlas $\mathcal{A}$ for M is a collection of charts whose domains cover M where each pair of charts is smoothly compatible.

A <u>smooth manifold</u> is (basically) a topological manifold together with a smooth atlas.

<u>Ex</u>: $M = S^1$



$\varphi(x,y) = x \qquad \psi(x,y) = y+3$

$\varphi^{-1}(t) = (t, \sqrt{1-t^2}) \qquad \psi^{-1}(t) = (\sqrt{1-(t-3)^2}, \, t-3)$

<u>Transition maps</u>: $\psi \circ \varphi^{-1} : (0,1) \longrightarrow (2,3)$

$$\psi \circ \varphi^{-1}(t) = \sqrt{1-t^2} + 3$$

$\varphi \circ \psi^{-1} : (2,3) \longrightarrow (0,1) \qquad t \longrightarrow \sqrt{1-(t-3)^2}$

Both of these are smooth ($\sqrt{\ }$ is not smooth at 0, but that's always (just) outside the domain of these fns.