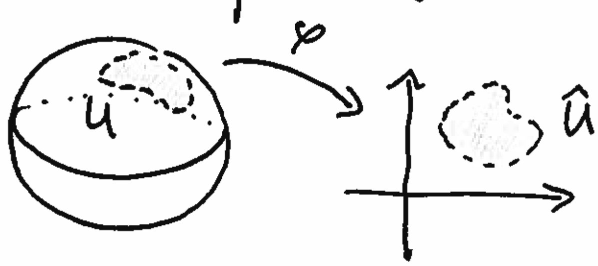
Lecture 2: Smooth Manifolds

Previously on Math 518:

Locally Euclidean: $\forall x \in X$ have open nbhds $U_x \overset{\text{homeomorphic}}{\cong} \mathbb{R}^n$.

Topological n-manifold: A Hausdorff topological space X with a countable basis which is locally Euclidean of dimension n.

Chart: (U, φ) where $\varphi : U \to \hat{U} \subseteq \mathbb{R}^n$ is a a homeomorphism form an open set in X to one in $\mathbb{R}^n$.



Survey results:

Green's Thm: 7

Stokes' Thm: 6

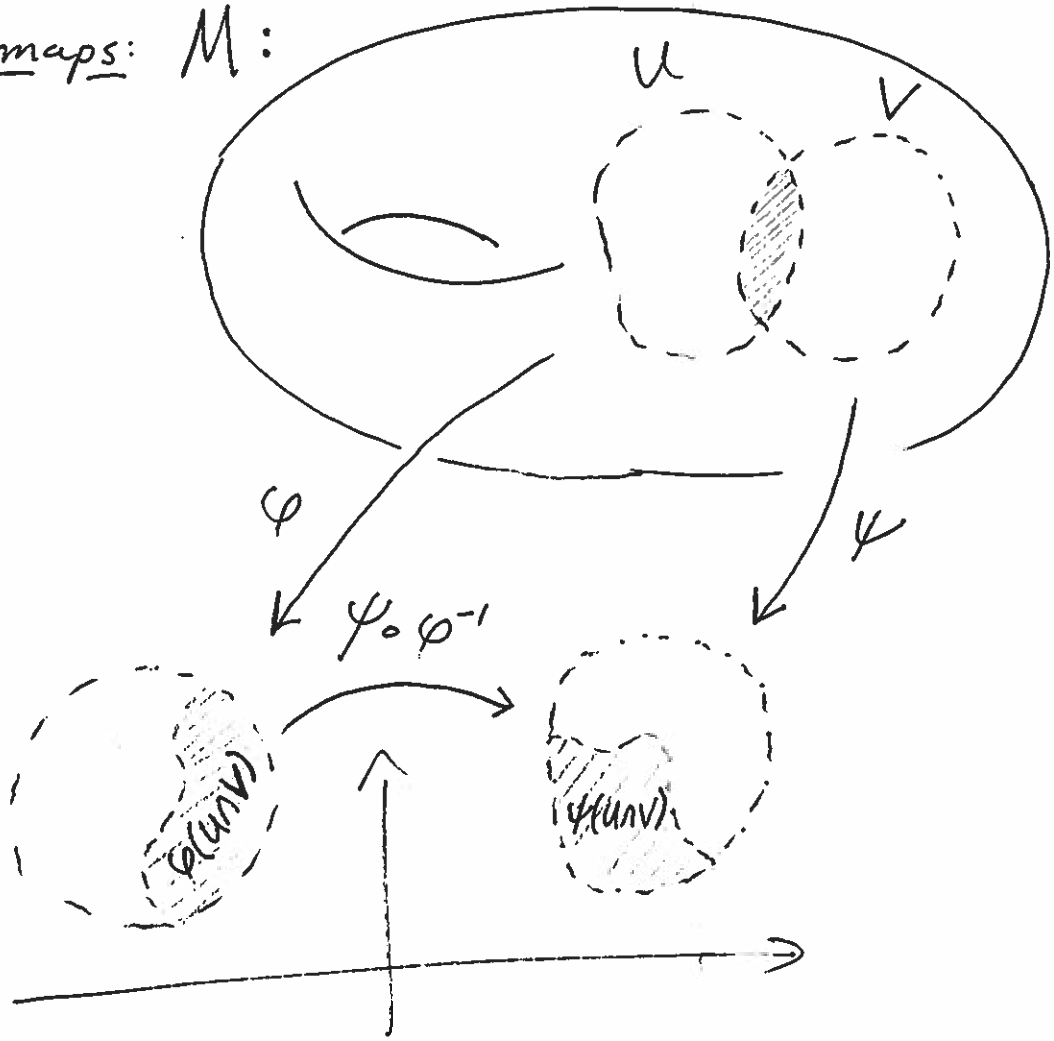
Divergence Thm: 3

Fubini's Thm: 2

Fund. Thm of Calc: 1

no answer / declined to state: 13

Transition maps: M:



The function

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

is a homeomorphism between open sets in $\mathbb{R}^n$.

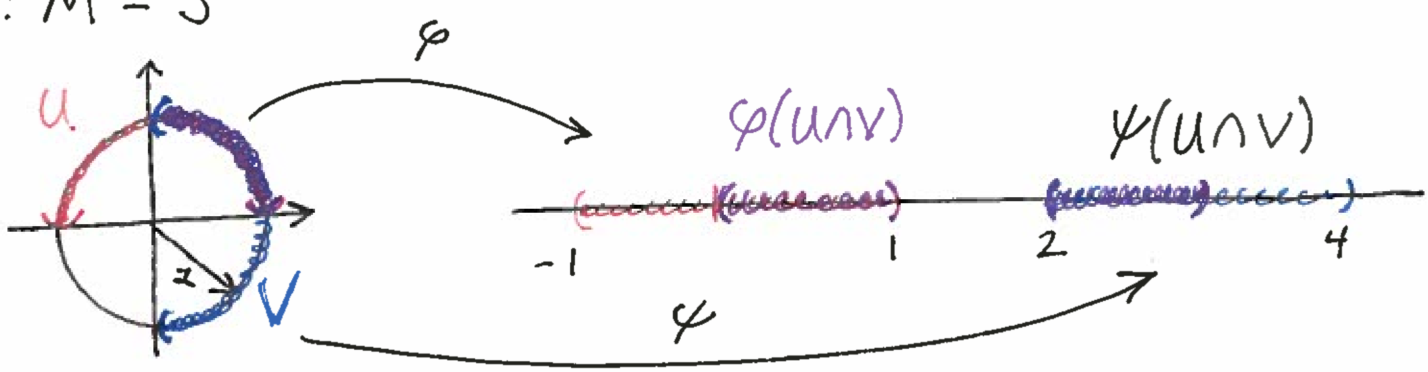
Charts (U, φ) and (V, ψ) are compatible if $\psi \circ \varphi^{-1}$ is a diffeomorphism, i.e. $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are both smooth.

A *smooth atlas* is a collection $\mathcal{A}$ of charts whose domains cover M where each pair is compatible.

A *smooth manifold* is a topological manifold together with a smooth atlas.

[Pause for questions; put in general context, stress importance.]

Ex: $M = S^1$



$$\varphi(x,y) = x \qquad \psi(x,y) = y+3$$

$$\varphi^{-1}(t) = (t, \sqrt{1-t^2}) \qquad \psi^{-1}(t) = \left(\sqrt{1-(t-3)^2}, t-3\right)$$

$$\psi\circ\varphi^{-1}: (0,1) \longrightarrow (2,3) \qquad t \longmapsto \sqrt{1-t^2}+3$$

$$\varphi\circ\psi^{-1}: (2,3) \longrightarrow (0,1) \qquad t \longmapsto \sqrt{1-(t-3)^2}$$

Both are smooth, even though $\sqrt{\cdot}$ is not smooth at 0.

Together with two similar charts ( + )
get a smooth atlas for S^1

[First HW is mostly getting a feel for charts + atlases.]

Notes: • The smooth atlas is a choice, not determined by the topology alone. ← called a smooth structure

• Some topological mflds have no smooth structures. [e.g. Kervaire's compact 10 mflds.]

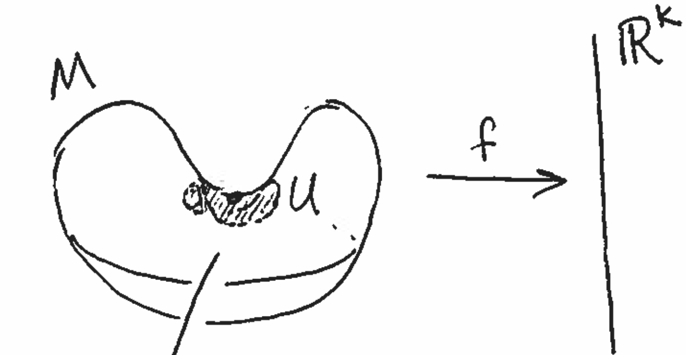
• Some top mflds have fund. different smooth structures; S^7 has 28, $\mathbb{R}^4$ has uncountably many.

[Top mflds of dim ≤ 3 have unique smooth strs.]

• Best to work with maximal atlases when defining smooth manifolds. That is, given $\mathcal{A}$ use $\bar{\mathcal{A}} = \{(V,\psi)$ compatible with every (U,φ) in $\mathcal{A}\}$

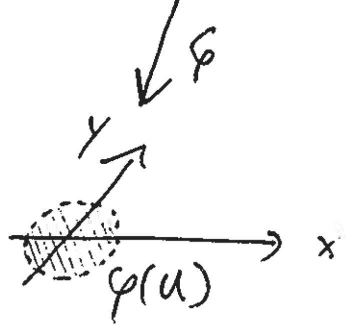
See [Lee, Prop 1.17].

<u>Def:</u> $(M, \mathcal{A})$ smooth n-manifold. A fn $f: M \to \mathbb{R}^k$ is smooth if for every $(U, \varphi) \in \mathcal{A}$ the fn $f \circ \varphi^{-1} : \varphi(U) \longrightarrow \mathbb{R}^k$ is smooth.



$\hat{f} = f \circ \varphi^{-1}(x, y) = x^2 - y^2$

coordinate representation



$f: S^1 \longrightarrow \mathbb{R} \quad f(x, y) = x + \sin y$

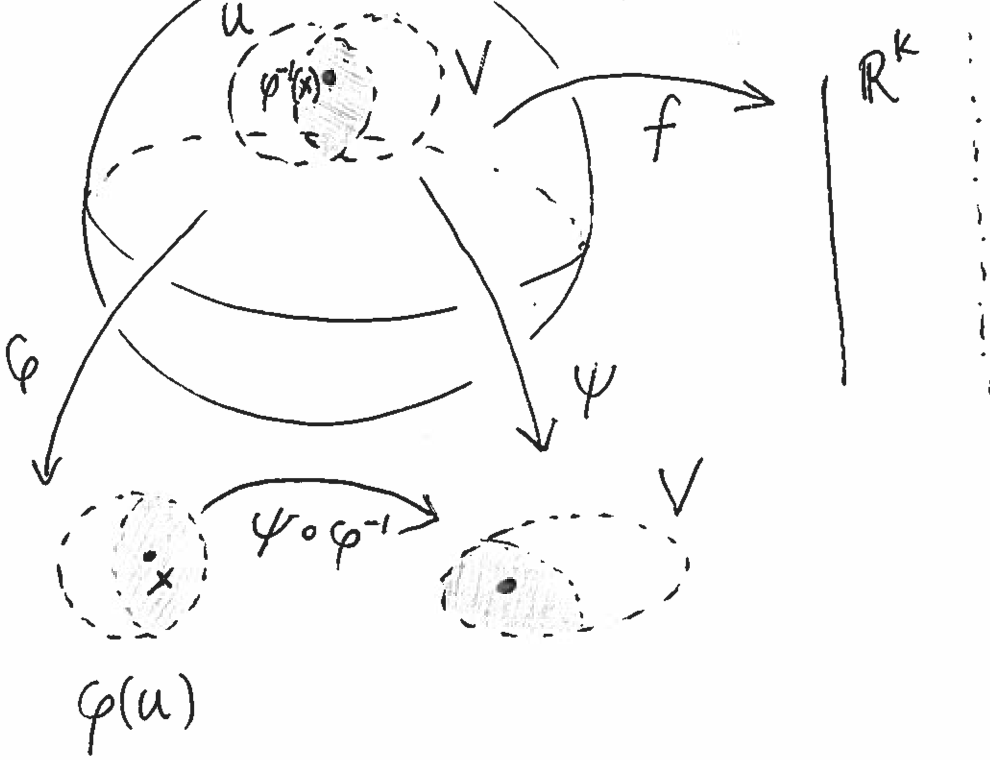
$f \circ \varphi^{-1}(t) = t + \sin\sqrt{1 - t^2}$

$f \circ \psi^{-1}(t) = \sqrt{1 - (t-3)^2} + \sin(t-3)$ ~~Example~~

<u>Lemma:</u> M smooth mfld, $f: M \to \mathbb{R}^k$. If every $p \in M$ is contained in a smooth chart $(\in \mathcal{A})$ where $\hat{f}$ is smooth, then f is smooth.

<u>Pf:</u> Given a smooth chart (U, φ) need to show $f \circ \varphi^{-1}$ is smooth. Smoothness is local, so focus on

$x \in \varphi(U)$. Let (V, ψ) be a smooth chart where $\varphi^{-1}(x) \in V$ and $f \circ \psi^{-1}$ is smooth.



Now, on $\varphi(U \cap V)$, we have

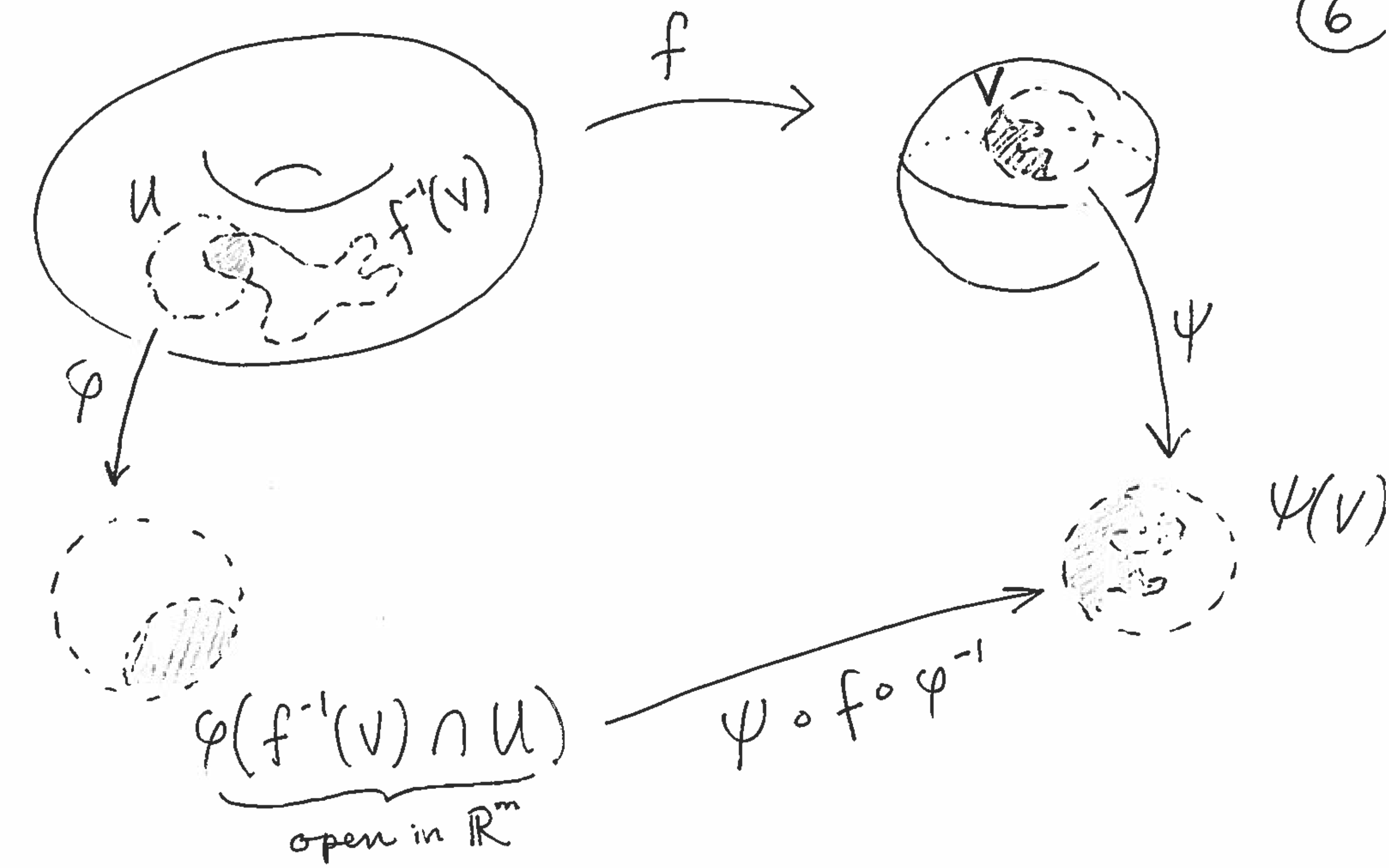
$$f \circ \varphi^{-1} = \underbrace{(f \circ \psi^{-1})}_{\text{smooth}} \circ \underbrace{(\psi \circ \varphi^{-1})}_{\text{smooth}}$$

and so $f \circ \varphi^{-1}$ is smooth on $\varphi(U \cap V) \ni x$. ∎

<u>Def</u>: A continuous $f: M \to N$ between two smooth mflds is smooth if for all smooth charts (U, φ) on M and (V, ψ) on N the fn

$$\psi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V) \cap U) \longrightarrow \psi(V)$$

is smooth.



Note: Equivalently, a fn $f: M \to N$ is smooth if $\forall p \in M$ there are smooth charts (U, φ) of M and (V, ψ) of M where

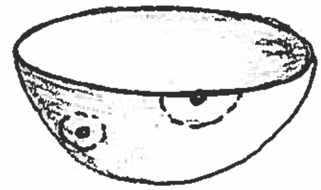
ⓐ $p \in U$ and $f(U) \subseteq V$

ⓑ $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth

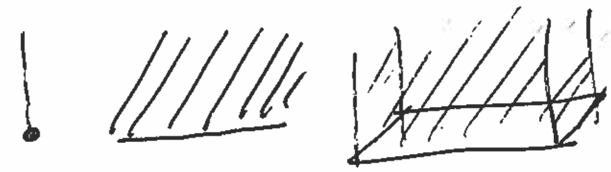
See [Lee, pgs 34-36.]

[Remark: In this version, no need to assume f is continuous.]

Manifolds with boundary:



Topological: M "reasonable" space where each pt has an open nbhd homeo to $\mathbb{R}^n$ or $\{\vec{x} \in \mathbb{R}^n \mid x_n \geq 0\}$



Smooth: A topological manifold with boundary with a smooth atlas of charts to $\mathbb{R}^n$ and $\mathbb{R}^n_{x_n \geq 0}$.

Techical pt: $H^n = \mathbb{R}^n_{x_n \geq 0}$ isn't open, so what does it mean for $f: H^n \to \mathbb{R}^n$ to be smooth?

A: There is an open set $U \supseteq H^n$ and a smooth fn $\bar{f}: U \to \mathbb{R}^n$ with $\bar{f}\big|_{H^n} = f$.